4. $y' = y^2 \sin x \quad \Rightarrow \quad \dfrac{dy}{dx} = y^2 \sin x \quad \Rightarrow \quad \dfrac{dy}{y^2} = \sin x\, dx \quad [y \neq 0] \quad \Rightarrow \quad \displaystyle\int \frac{dy}{y^2} = \int \sin x\, dx \quad \Rightarrow$

$-\dfrac{1}{y} = -\cos x + C \quad \Rightarrow \quad \dfrac{1}{y} = \cos x - C \quad \Rightarrow \quad y = \dfrac{1}{\cos x + K}$, where $K = -C$. $y = 0$ is also a solution.

10. $\dfrac{dy}{dx} = \dfrac{y \cos x}{1 + y^2}$, $y(0) = 1$. $(1 + y^2)\, dy = y \cos x\, dx \quad \Rightarrow \quad \dfrac{1 + y^2}{y} dy = \cos x\, dx \quad \Rightarrow \quad \int \left(\frac{1}{y} + y\right) dy = \int \cos x\, dx \quad \Rightarrow$
$\ln|y| + \frac{1}{2}y^2 = \sin x + C$. $y(0) = 1 \quad \Rightarrow \quad \ln 1 + \frac{1}{2} = \sin 0 + C \quad \Rightarrow \quad C = \frac{1}{2}$, so $\ln|y| + \frac{1}{2}y^2 = \sin x + \frac{1}{2}$. We cannot solve explicitly for y.

16. $\dfrac{dy}{dx} = \dfrac{y^2}{x^3}$, $y(1) = 1$. $\displaystyle\int \frac{dy}{y^2} = \int \frac{dx}{x^3} \quad \Rightarrow \quad -\frac{1}{y} = -\frac{1}{2x^2} + C$. $y(1) = 1 \quad \Rightarrow \quad -1 = -\frac{1}{2} + C \quad \Rightarrow \quad C = -\frac{1}{2}$.

So $\dfrac{1}{y} = \dfrac{1}{2x^2} + \dfrac{1}{2} = \dfrac{2 + 2x^2}{2 \cdot 2x^2} \quad \Rightarrow \quad y = \dfrac{2x^2}{x^2 + 1}$.

29. $\dfrac{dP}{dt} = k(M - P) \quad \Leftrightarrow \quad \displaystyle\int \frac{dP}{P - M} = \int (-k)\, dt \quad \Leftrightarrow \quad \ln|P - M| = -kt + C \quad \Leftrightarrow \quad |P - M| = e^{-kt + C} \quad \Leftrightarrow$

$P - M = Ae^{-kt}$ $[A = \pm e^C] \quad \Leftrightarrow \quad P = M + Ae^{-kt}$. If we assume that performance is at level 0 when $t = 0$, then $P(0) = 0 \quad \Leftrightarrow \quad 0 = M + A \quad \Leftrightarrow \quad A = -M \quad \Leftrightarrow \quad P(t) = M - Me^{-kt}$. $\displaystyle\lim_{t \to \infty} P(t) = M - M \cdot 0 = M$.

7. (a) Our assumption is that $\dfrac{dy}{dt} = ky(1-y)$, where y is the fraction of the population that has heard the rumor.

(b) Using the logistic equation (1), $\dfrac{dP}{dt} = kP\left(1 - \dfrac{P}{K}\right)$, we substitute $y = \dfrac{P}{K}$, $P = Ky$, and $\dfrac{dP}{dt} = K\,\dfrac{dy}{dt}$,

to obtain $K\,\dfrac{dy}{dt} = k(Ky)(1-y) \quad \Leftrightarrow \quad \dfrac{dy}{dt} = ky(1-y)$, our equation in part (a).

Now the solution to (1) is $P(t) = \dfrac{K}{1 + Ae^{-kt}}$, where $A = \dfrac{K - P_0}{P_0}$.

We use the same substitution to obtain $Ky = \dfrac{K}{1 + \dfrac{K - Ky_0}{Ky_0}e^{-kt}} \quad \Rightarrow \quad y = \dfrac{y_0}{y_0 + (1-y_0)e^{-kt}}$.

Alternatively, we could use the same steps as outlined in "The Analytic Solution," following Example 2.

(c) Let t be the number of hours since 8 A.M. Then $y_0 = y(0) = \frac{80}{1000} = 0.08$ and $y(4) = \frac{1}{2}$, so

$\dfrac{1}{2} = y(4) = \dfrac{0.08}{0.08 + 0.92e^{-4k}}$. Thus, $0.08 + 0.92e^{-4k} = 0.16$, $e^{-4k} = \frac{0.08}{0.92} = \frac{2}{23}$, and $e^{-k} = \left(\frac{2}{23}\right)^{1/4}$, so

$y = \dfrac{0.08}{0.08 + 0.92(2/23)^{t/4}} = \dfrac{2}{2 + 23(2/23)^{t/4}}$. Solving this equation for t, we get

$$2y + 23y\left(\frac{2}{23}\right)^{t/4} = 2 \quad \Rightarrow \quad \left(\frac{2}{23}\right)^{t/4} = \frac{2-2y}{23y} \quad \Rightarrow \quad \left(\frac{2}{23}\right)^{t/4} = \frac{2}{23}\cdot\frac{1-y}{y} \quad \Rightarrow \quad \left(\frac{2}{23}\right)^{t/4-1} = \frac{1-y}{y}.$$

It follows that $\dfrac{t}{4} - 1 = \dfrac{\ln[(1-y)/y]}{\ln\frac{2}{23}}$, so $t = 4\left[1 + \dfrac{\ln((1-y)/y)}{\ln\frac{2}{23}}\right]$.

When $y = 0.9$, $\dfrac{1-y}{y} = \frac{1}{9}$, so $t = 4\left(1 - \dfrac{\ln 9}{\ln\frac{2}{23}}\right) \approx 7.6$ h or 7 h 36 min. Thus, 90% of the population will have heard the rumor by 3:36 P.M.

Problem Set 24

Differential Equations Handout
Problems: 6, 8, 9, 11

April 26, 2006

Problem 6

Part A

As the problem states, the instantaneous rate of increase of the balance from interest is 4% or .04. This we see that

$$\frac{dM}{dt} = .04M(t)$$

where $M(0) = 2000$ since start by making an initial deposit of \$2000

Part B

Solving the differential equation we see that

$$\frac{dM}{M} = .04dt \Rightarrow \int \frac{dM}{M} = \int .04dt \Rightarrow \ln M = .04t + c \Rightarrow M = e^{.04t+c} \Rightarrow M(t) = Ce^{.04t}$$

Given the initial condition that $2000 = M(0) = Ce^0 = C$ we see that $C = 2000$ and our final solution is

$$M(t) = 2000e^{.04t}$$

Part C

Now we are adding 1000 per year continuously. So our differential equation becomes:

$$\frac{dM}{dt} = .04M + 1000$$

with the initial condition that $M(0) = 2000$

Part D

We already solved the equation that $\frac{dM}{dt} = .04M$ Now that we have an extra 1000 dollars per year, we will guess a similar solution for $M(t)$ and solve for the constant. We guess that

$$M(t) = Ce^{.04t} + k \Rightarrow \frac{dMt}{dt} = (.04)(Ce^{.04t} = .04M(t) - .04k = .04M(t) + 1000$$

Thus we see that $k = -25000$ and that

$$M(t) = Ce^{.04t} - 25000 \Rightarrow 2000 = M(0) = c - 25000$$

So we see our final solution is

$$M(t) = 27000e^{.04t} - 25000$$

Problem 8

Part A

We are given that the growth is proportional to the population so our differential equation will be:

$$\frac{dM}{dt} = kM(t)$$

when $M(t) = 5000$ we see that $\frac{dM}{dt} = 250$. This we see that $k = \frac{1}{20}$.

Part B

For this we will actually need to solve the differential equation. First lets write it out:

$$\frac{dB(t)}{dt} = kB(t)$$

Solving this in the same manner as problem 6 yields

$$B(t) = Ce^{kt}$$

At $t = 0$ we have that $B(t) = 600$ which tells us that $C = 600$. Then that 10 we have that

$$800 = 600e^{10k} \Rightarrow \frac{4}{3} = e^{10k} \Rightarrow k = \frac{1}{10}\ln\left(\frac{4}{3}\right)$$

Problem 9

Part A

So our differential equation is just

$$\frac{dP(t)}{dt} = .03P(t) - 6000$$

where the 6000 is negative since people are leaving.

Part B

Similarly to problem 8, our solution will look like

$$P(t) = Ce^{.03t} - 200,000$$

where are initial condition is that $C = P(0) = 3,000,000$ so our final solution is

$$P(t) = 3,200,000e^{.03t} - 200,000$$

Problem 11

Part A

It is being replenished at the same rate as the paint is taken out, so if 2 liters per hour are leaving and 20% of the refilling is done by white paint, then we have that the amount of white paint going in is .4 liters per hour. We will also us white paint at the rate of $2\frac{w(t)}{10}$ since that is the fraction of white paint times the rate at which we use it. Thus we get that

$$\frac{dw(t)}{dt} = .4 - \frac{w(t)}{5}$$

given that $w(0) = 0$. Solving this equation we get that:

$$w(t) = Ce^{-.2t} + .4$$

given the initial condition that $w(0) = 0$ we have that $C = -.4$. The solution to this will be

$$w(t) = e^{-.2t} + 2$$

although this is not neccesary. Basically, the amount of white paint will asymptotically approach two liters.

Part B

This is similar to the white paint version above. The differential equation will just be

$$\frac{db(t)}{dt} = 1.6 - \frac{b(t)}{5}$$

with $b(0) = 10$. This will asymptotically approach 8 liters from above.