34. (a) Use 1 billion dollars as the x-unit and 1 day as the t-unit. Initially, there is \$10 billion of old currency in circulation, so all of the \$50 million returned to the banks is old. At time t, the amount of new currency is $x(t)$ billion dollars, so $10 - x(t)$ billion dollars of currency is old. The fraction of circulating money that is old is $[10 - x(t)]/10$, and the amount of old currency being returned to the banks each day is $\dfrac{10 - x(t)}{10} \cdot 0.05$ billion dollars. This amount of new currency per day is introduced into circulation, so $\dfrac{dx}{dt} = \dfrac{10 - x}{10} \cdot 0.05 = 0.005(10 - x)$ billion dollars per day.

(b) $\dfrac{dx}{10 - x} = 0.005\,dt \;\Rightarrow\; \dfrac{-dx}{10 - x} = -0.005\,dt \;\Rightarrow\; \ln(10 - x) = -0.005t + c \;\Rightarrow\; 10 - x = Ce^{-0.005t}$, where $C = e^c \;\Rightarrow\; x(t) = 10 - Ce^{-0.005t}$. From $x(0) = 0$, we get $C = 10$, so $x(t) = 10(1 - e^{-0.005t})$.

(c) The new bills make up 90% of the circulating currency when $x(t) = 0.9 \cdot 10 = 9$ billion dollars.
$9 = 10(1 - e^{-0.005t}) \;\Rightarrow\; 0.9 = 1 - e^{-0.005t} \;\Rightarrow\; e^{-0.005t} = 0.1 \;\Rightarrow\; -0.005t = -\ln 10 \;\Rightarrow\;$
$t = 200 \ln 10 \approx 460.517$ days ≈ 1.26 years.

19. (a) Using $A = A_0\left(1 + \dfrac{r}{n}\right)^{nt}$ with $A_0 = 3000$, $r = 0.05$, and $t = 5$, we have:

$\quad$ (i) Annually: $n = 1$; $\qquad A = 3000\left(1 + \frac{0.05}{1}\right)^{1 \cdot 5} = \$3828.84$

$\quad$ (ii) Semiannually: $n = 2$; $\quad A = 3000\left(1 + \frac{0.05}{2}\right)^{2 \cdot 5} = \$3840.25$

$\quad$ (iii) Monthly: $n = 12$; $\qquad A = 3000\left(1 + \frac{0.05}{12}\right)^{12 \cdot 5} = \$3850.08$

$\quad$ (iv) Weekly: $n = 52$; $\qquad A = 3000\left(1 + \frac{0.05}{52}\right)^{52 \cdot 5} = \$3851.61$

$\quad$ (v) Daily: $n = 365$; $\qquad A = 3000\left(1 + \frac{0.05}{365}\right)^{365 \cdot 5} = \$3852.01$

$\quad$ (vi) Continuously: $\qquad A = 3000e^{(0.05)5} = \$3852.08$

(b) $dA/dt = 0.05A$ and $A(0) = 3000$.

Problem Set 25

Differential Equations Supplement
Problems: 1, 2, 3, 5, 6, 7 on pages 1014-1016

April 28, 2006

Problem 1

Part A

We are given the differential equation

$$\frac{dy}{dt} = 4y - 8$$

We can see that the derivative is 0 at y=2. Below this line the derivative is negative and above it positive, thus its an unstable equilibrium and the solutions will diverge away from it. We are given 3 initial conditions: $y(0) = 0$, $y(0) = -1$, and $y(0) = 3$. These are plotted below on a slope field.
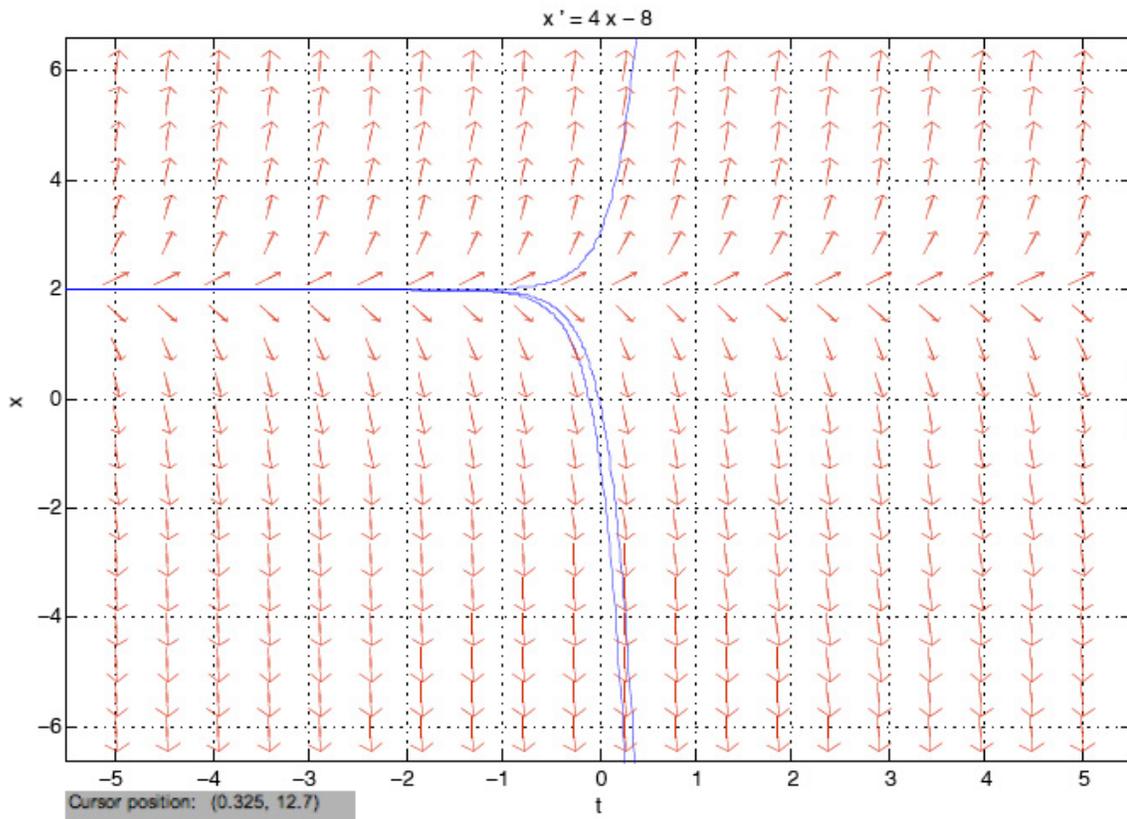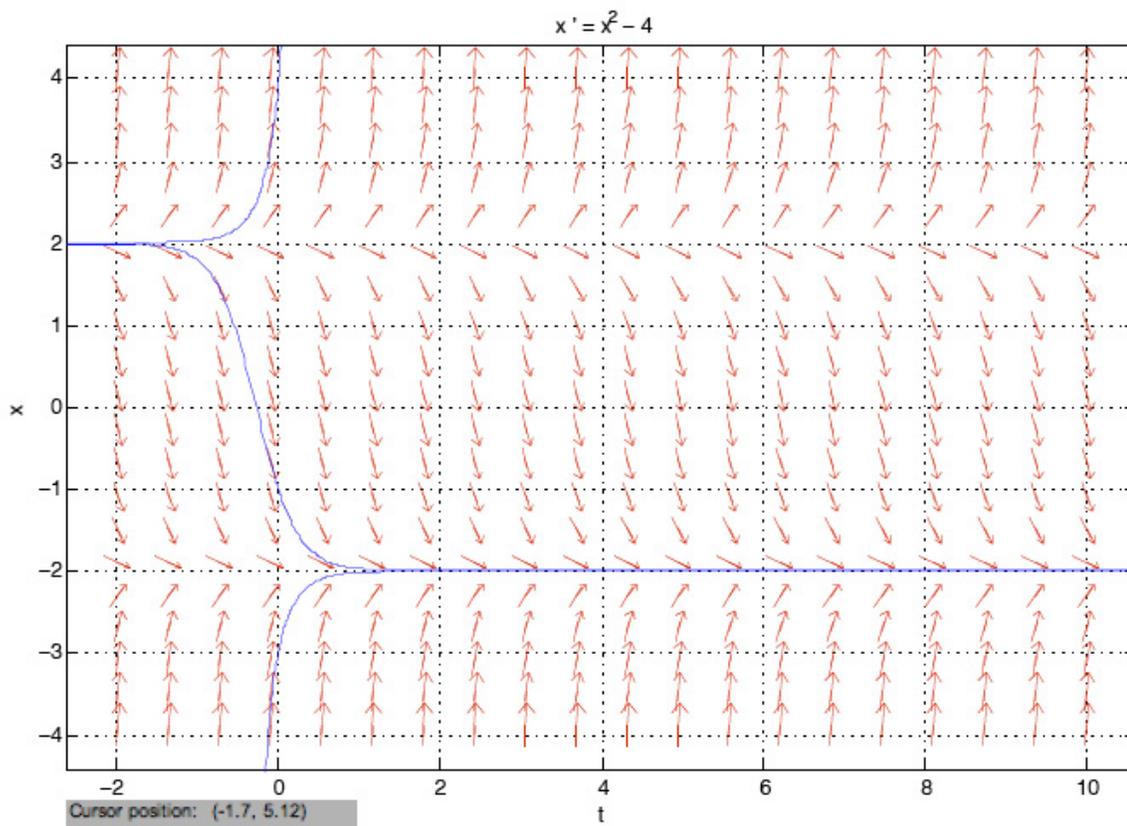
Part B

We are given

$$\frac{dy}{dt} = y^2 - 4$$

We see that the derivative is 0 at y=-2, 2. Looking at the sign of the derivative we can see that y=2 is unstable and y=-2 is stable. The initial conditions of $y(0) = -1$, $y(0) = -3$, and $y(0) = 4$ are plotted below on the slope field.

$$x' = x^2 - 4$$

Cursor position: (-1.7, 5.12)

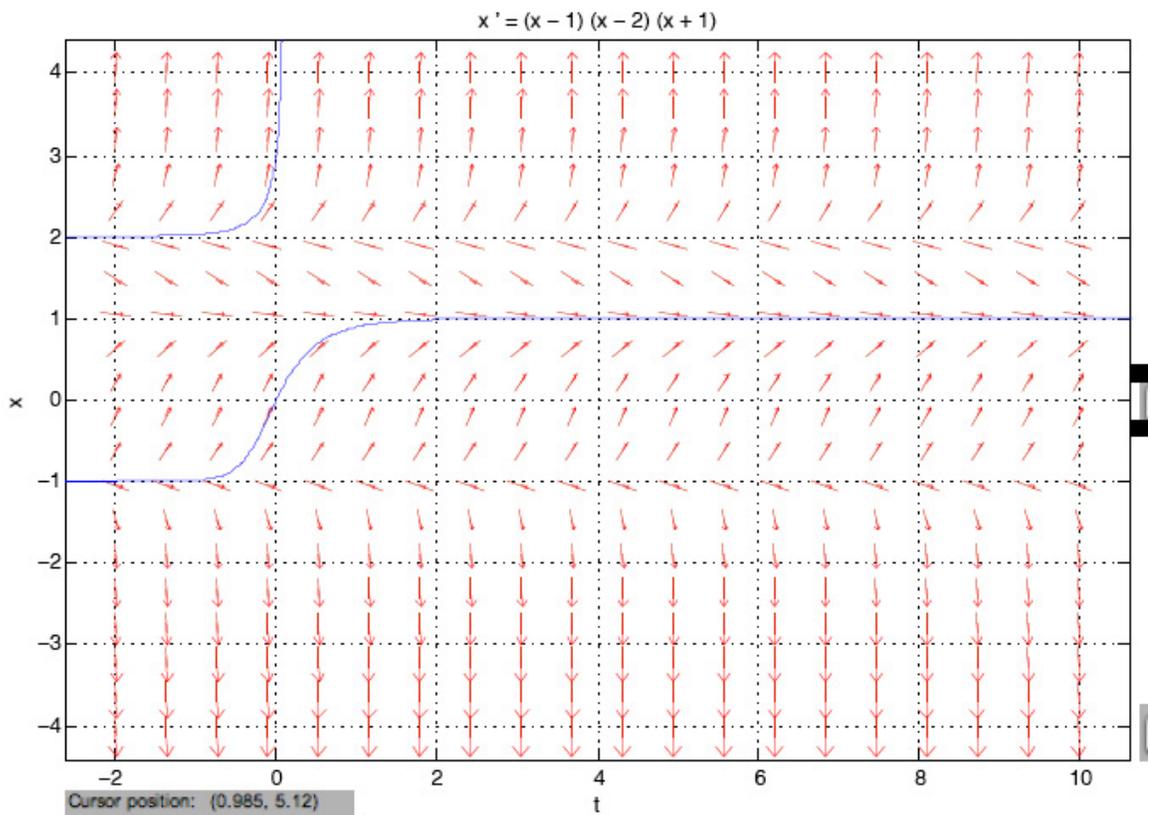## Part C

We are given that

$$\frac{dy}{dt} = (y-1)(y-2)(y+1)$$

We can see that the derivative is 0 at $y = -1, 1, 2$ so all three of these are equilibria. Looking at the sign of the derivative we can see that y=1 is a stable equilibrium while the other two are unstable. The curves belonging to the initial conditions of $y(0) = 0$ and $y(0) = 3$ are graphed below with the slope field.

x ' = (x − 1) (x − 2) (x + 1)
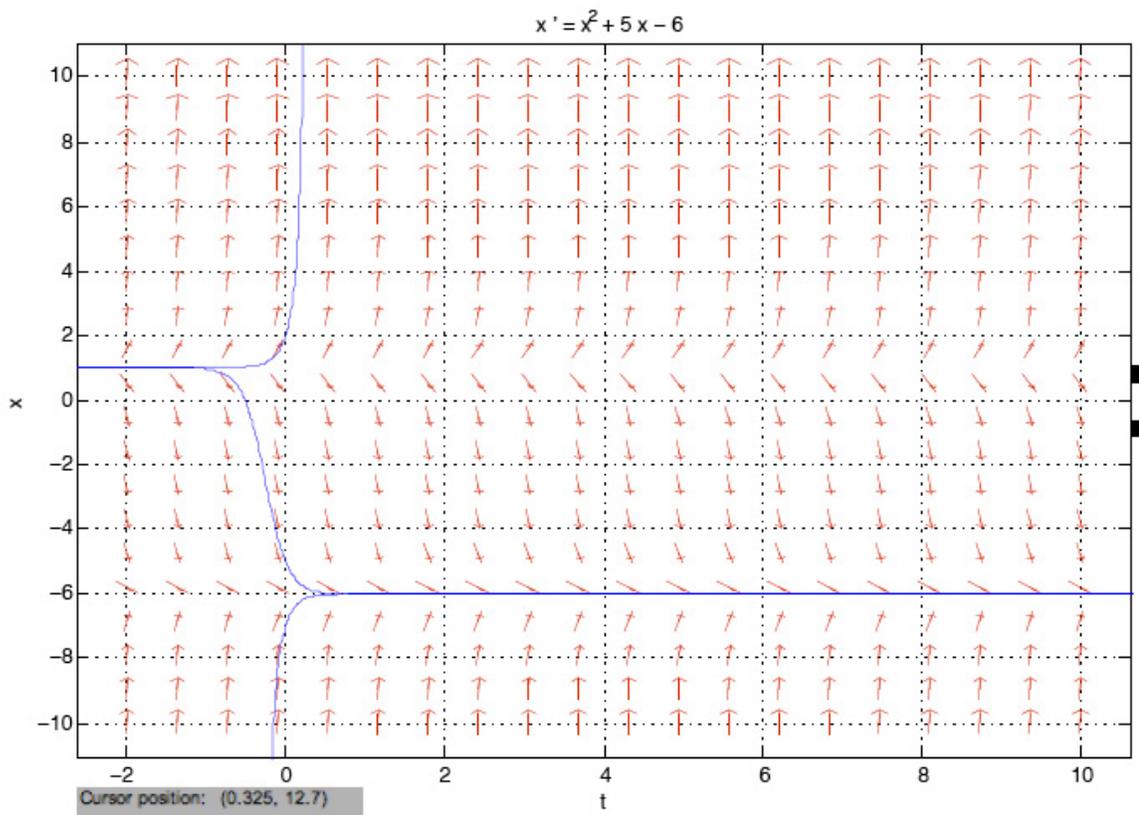
Cursor position: (0.985, 5.12)

Part D

We are given that

$$\frac{dy}{dt} = y^2 + 5y - 6 = (y + 6)(y - 1)$$

where by factoring we can see that the derivative is 0 at $y = -6$ and $y = 1$, where y=-6 will be stable and y=1 will be unstable. Again, we've plotted the three intiial conditions (y(0)=-5, y(0) = -7, and y(0)=2) below on the slope field.

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$$x' = x^2 + 5x - 6$$

Cursor position: (0.325, 12.7)

Problem 2

Part A

We can see that the function asymptotically approaches y=3, so we need our derivative by 0 there. We also need it to go towards 3 from above and below, so a decent answer would be

$$\frac{dy}{dt} = 3 - y$$

Part B

We still have a line at 0, its just that it diverges on both sides, this above we want the derivative to positive, and below y=3, negative. Thus a decent guess would be

$$\frac{dy}{dt} = y - 3$$

Part C

We want it to diverge from y=t and converge to y=2. Thus a decent solution would be

$$\frac{dy}{dt} = y(2 - y)$$

As we can see, when y is greater than 2 the derivative is negative, between 0 and 2 it is positive, and lastly when y is negative is negative again.

Part D

We want it to converge to -2 and diverge from 2, and $\frac{dy}{dt}$ to be 0 at y=2 and -2. Thus a solution of the form

$$\frac{dy}{dt} = (y - 2)(y + 2)$$

since this will be positive if $|y| > 2$ and negative in between.

Problem 3

Part A

Since the graphs converge to $y = 3$ it will be a stable equilibrium.

Part B

We have an equilibrium again ay $y = 3$, but this time it is unstable since the graphs curve away.

Part C

The graphs converge to $y = 2$ and diverge from $y = 0$ so $y = 2$ is a stable equilibrium and $y = 0$ is unstable.

Part D

We have equilibria at $y = \pm 2$ where $y = 2$ is unstable and $y = -2$ is stable.

Problem 5

We are given the differential equation

$$\frac{dP}{dt} = .02P - N$$

Part A

We just need to make sure that the derivative is positive, otherwise the population would decrease and die out. We just need the derivative to be positive at $t = 0$ and we'll ensure an increasing population forever. Thus we say that

$$0 < .02P(0) - N \Rightarrow N < 2$$

Thus the yearly exodus must be less than 2000 people per year.

Part B

Similarly we let

$$0 < .02P(0) - 1 \Rightarrow P(0) > 50$$

So the population must be at least 50,000 people to start with.

Problem 6

We are given that

$$\frac{dy}{dt} = y^2 - 1 = (y - 1)(y + 1)$$

From this we can determine that the slope will be negative for $-1 < y < 1$ and positive everywhere else. So we can see that $y = -1$ will be a stable equilibrium and $y = +1$ will be unstable. This rules out everything but option D, which is the answer. Options A, B, and E all start at y=0 but don't converge to y=-1 as they should. Option C converges to y=1 which is unstable. Option D works because it converges to y=-1, our stable equilibrium.

Problem 7

We are given that

$$\frac{dx}{dt} = x^2(2 - x)$$

where $x(0) = .5$

Looking at the differential equation, we see that the derivative is positive at x=.5, so this rules out the top two. Between 0 and 2 the derivative is positive, and above 2 the derivative is negative so it is a stable equilibrium. This rules out the bottom right since it does not approach x=2 as it should. This leaves the bottom left as the best answer.