

Problem Set 26

First-Order Differential Eq. Handout - Exercises 1, 2, 3

May 9, 2006

Exercises 1

We are given that

$$y' - 4xy = x$$

We'll use an integrating factor and multiply through by a factor of e^{-2x^2}

$$\frac{dy}{dx}e^{-2x^2} - 4xe^{-2x^2}y = xe^{-2x^2} \Rightarrow d(e^{-2x^2}y) = xe^{-2x^2}dx \Rightarrow e^{-2x^2}y = \frac{-1}{4}e^{-2x^2} + C \Rightarrow y = \frac{-1}{4} + Ce^{2x^2}$$

Exercise 2

We are given that

$$xy' + y = x^3$$

This is already in a nice form, so we just need to group it nicely.

$$\frac{d(xy)}{dx} = x^3 \Rightarrow xy = \frac{x^4}{4} + C \Rightarrow y = \frac{x^3}{4} + \frac{C}{x}$$

Exercise 3

We are given that

$$y' + y = e^x$$

We'll multiply through by our integrating factor

$$e^x y' + e^x y = e^{2x} \Rightarrow \frac{d(e^x y)}{dx} = e^{2x} \Rightarrow e^x y = \frac{1}{2}e^{2x} + C \Rightarrow y = \frac{1}{2}e^x + Ce^{-x}$$

We are also given that $y(0) = 6$ which tells us that

$$\frac{1}{2} + C = 6 \Rightarrow C = 11/2$$

So our final answer is that

$$y = \frac{1}{2}e^x + \frac{11}{2}e^{-x}$$

Problem Set 26

Differential Equation Handout - Exercises 5, 7, 10, 12

May 1, 2006

Exercise 5

Part A

We have that the rate at which people are becoming infected is proportional to the product of infected people and healthy people, thus our differential equation becomes

$$\frac{I(t)}{dt} = \alpha (N - I(t)) I(t)$$

We know that $0 \leq I(t) \leq N$ which means the right side is positive, and people are getting infected, which means that $\frac{dI(t)}{dt}$ is positive and thus $\alpha > 0$.

Part B

The number of infected people is always increasing with time, and in the long run everyone will get sick. Essentially $I(t)$ will asymptotically approach N .

Exercise 7

We have that he starts by taking 100,000 to start with and pays it back at 12,000 per year. He also gets charged a continuous interest rate of 7.25%. This we see that our differential equation will be

$$\frac{dB(t)}{dt} = .0725B(t) - 12000$$

where $B(0) = 100,000$

Exercise 10

Part A

- For $y(0) = 0$ we have that $\frac{dy}{dx} = 0$ and the function will stay there for all time. So $y(x) = 0$

- For $y(0) = .01$ the derivative is positive, although starting very small, the function will increase without bound.
- For $y(0) = -.01$ the derivative is positive since it is y^2 so we will have that y will asymptotically approach 0 as x goes to infinity.

Part B

The solution is unstable because if we increase y by a bit, it will increase even more.

Part C

Separating the variables we have that

$$\frac{dy}{y^2} = dx \Rightarrow \frac{1}{y} = -x + c \Rightarrow y = \frac{-1}{x + C}$$

we are given that $y(0) = 1 \Rightarrow C = -1$ and we find that

$$y(x) = \frac{1}{1 - x}$$

Part D

As x goes to 1 from the left that $y(x)$ will diverge to positive infinity.

Exercise 12

We are given that

$$y'' + y' = -x^2$$

We know that $-x^2 < 0$ since x^2 must be positive and then we are negating it. The function cannot be both increasing and concave up because the former implies that $y' > 0$ and the later implies that $y'' > 0$. Thus on the left side we are adding together two quantities that are greater than 0 to equal a negative quantity, which is a contradiction. Thus it can't be both increasing and concave up.