**1.** (a) $dx/dt = -0.05x + 0.0001xy$. If $y = 0$, we have $dx/dt = -0.05x$, which indicates that in the absence of $y$, $x$ declines at a rate proportional to itself. So $x$ represents the predator population and $y$ represents the prey population. The growth of the prey population, $0.1y$ (from $dy/dt = 0.1y - 0.005xy$), is restricted only by encounters with predators (the term $-0.005xy$). The predator population increases only through the term $0.0001xy$; that is, by encounters with the prey and not through additional food sources.

(b) $dy/dt = -0.015y + 0.00008xy$. If $x = 0$, we have $dy/dt = -0.015y$, which indicates that in the absence of $x$, $y$ would decline at a rate proportional to itself. So $y$ represents the predator population and $x$ represents the prey population. The growth of the prey population, $0.2x$ (from $dx/dt = 0.2x - 0.0002x^2 - 0.006xy = 0.2x(1 - 0.001x) - 0.006xy$), is restricted by a carrying capacity of 1000 [from the term $1 - 0.001x = 1 - x/1000$] and by encounters with predators (the term $-0.006xy$). The predator population increases only through the term $0.00008xy$; that is, by encounters with the prey and not through additional food sources.

**2.** (a) $dx/dt = 0.12x - 0.0006x^2 + 0.00001xy$. $dy/dt = 0.08y + 0.00004xy$.

The $xy$ terms represent encounters between the two species $x$ and $y$. An increase in $y$ makes $dx/dt$ (the growth rate of $x$) larger due to the positive term $0.00001xy$. An increase in $x$ makes $dy/dt$ (the growth rate of $y$) larger due to the positive term $0.00004xy$. Hence, the system describes a cooperation model.

(b) $dx/dt = 0.15x - 0.0002x^2 - 0.0006xy = 0.15x(1 - x/750) - 0.0006xy$.

$dy/dt = 0.2y - 0.00008y^2 - 0.0002xy = 0.2y(1 - y/2500) - 0.0002xy$.

The system shows that $x$ and $y$ have carrying capacities of 750 and 2500. An increase in $x$ reduces the growth rate of $y$ due to the negative term $-0.0002xy$. An increase in $y$ reduces the growth rate of $x$ due to the negative term $-0.0006xy$. Hence, the system describes a competition model.

# Problem Set 28

Differential Equation Handout - Exercises 15, 17, 18

May 8, 2006

## Problem 15

We are given that

$$x' = ax - bx^2 - cxy \qquad y' = -dy + exy$$

### Part A

We know that all the constants are positive, so we see that x' decreases as y increases, so he might be the prey. Similarly, the population of y increases as x increases so y must be the predator.

### Part B

If we have no predators, we set $y = 0$. This tells us that $y' = 0$ so we don't spontaneously generated and predators and that

$$x' = ax - bx^2$$

The population of x will approach the point where $x' = 0$ which is $x = a/b$ (since we know $x \neq 0$. Thus we expect a long run prey population of $a/b$.

If on the other hand we start with some predators and no prey, we can see immediately that $x' = 0$ and $y' = -dy$. Since we know both $d$ and $y$ are positive, we can see that the derivative will be negative, and the population of predators will asymptotically approach 0.

### Part C

We will get an equilibrium when $x' = y' = 0$. Solving the second equation assuming $y \neq 0$ yields

$$0 = -dy + exy \Rightarrow x = \frac{d}{e}$$

Now we can plug this into the first equation

$$0 = ax - bx^2 - cxy \Rightarrow y = \frac{a - bx}{c} = \frac{a - bd/e}{c} = \frac{a}{c} - \frac{db}{ce}$$

Putting these two coordinates together we do get the solution

$$(x, y) = \left(\frac{d}{e}, \frac{a}{c} - \frac{bd}{ce}\right)$$

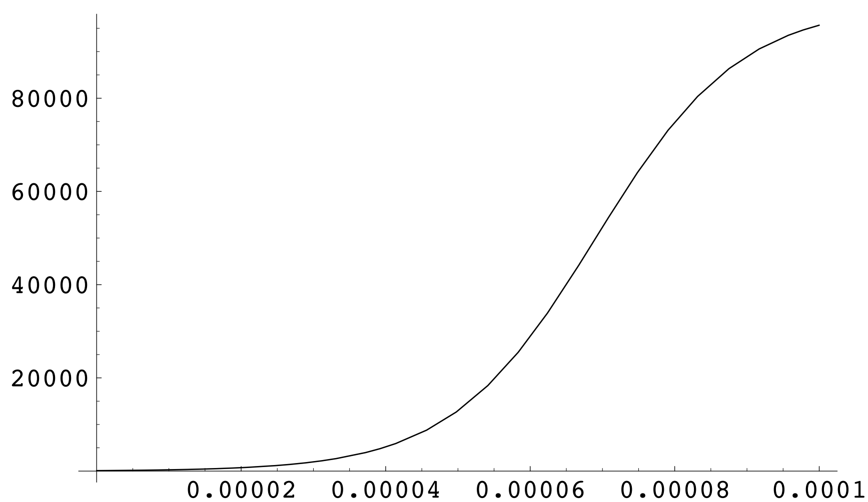
### Part D

Qualitatively we can describe what the constants mean in terms of population. The constant $a$ is a measure possibly of how fast the prey reproduce and $b$ maybe a measure of the detrimental effects of overcrowding. If we set $y = 0$ we see that this will still give us an equilibrium population for x, which makes sense since we'll balance out overcrowding and reproduction. The constant $c$ is just a measure of the detriment to x from the predation by $y$, basically how many prey a predator will kill. This is attached to the term $xy$ since an increased $x$ means the prey are easier to find and an increased $y$ means there's more predators looking. The $d$ term which causes $y'$ to decrease with increased y represents the competition of the predators with one another. If there are more predators then each will get less to eat. Lastly $e$ represents the benefit $y$ gets from eating $x$, basically how often prey is caught.

## Problem 17

### Part A

The population will approach the value such that $A' = 0$, which is at $A = 100,000$. It will try and climb exponentially, but then end up asymptotically approach towards the end, so it will look like the letter s, vaguely. The solution for $A(t)$ is graphed below.

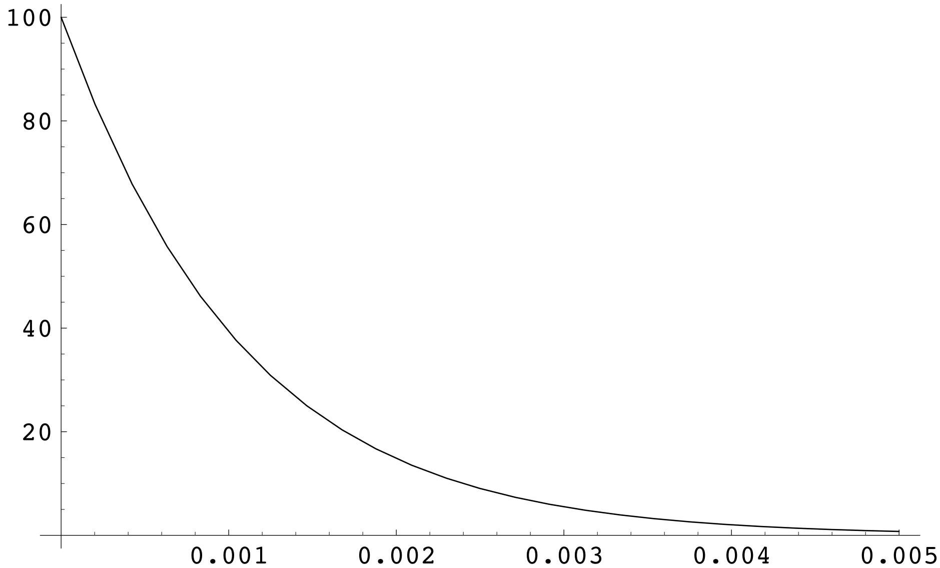


**Part B**

We are given that

$$L' = -L(1000 + L)$$

We can see that $L'$ will be negative unless $-1000 < L < 0$, so if we start with a positive number of ladybugs, it will asymptotically approach 0. The solution is graphed below just to see the picture.

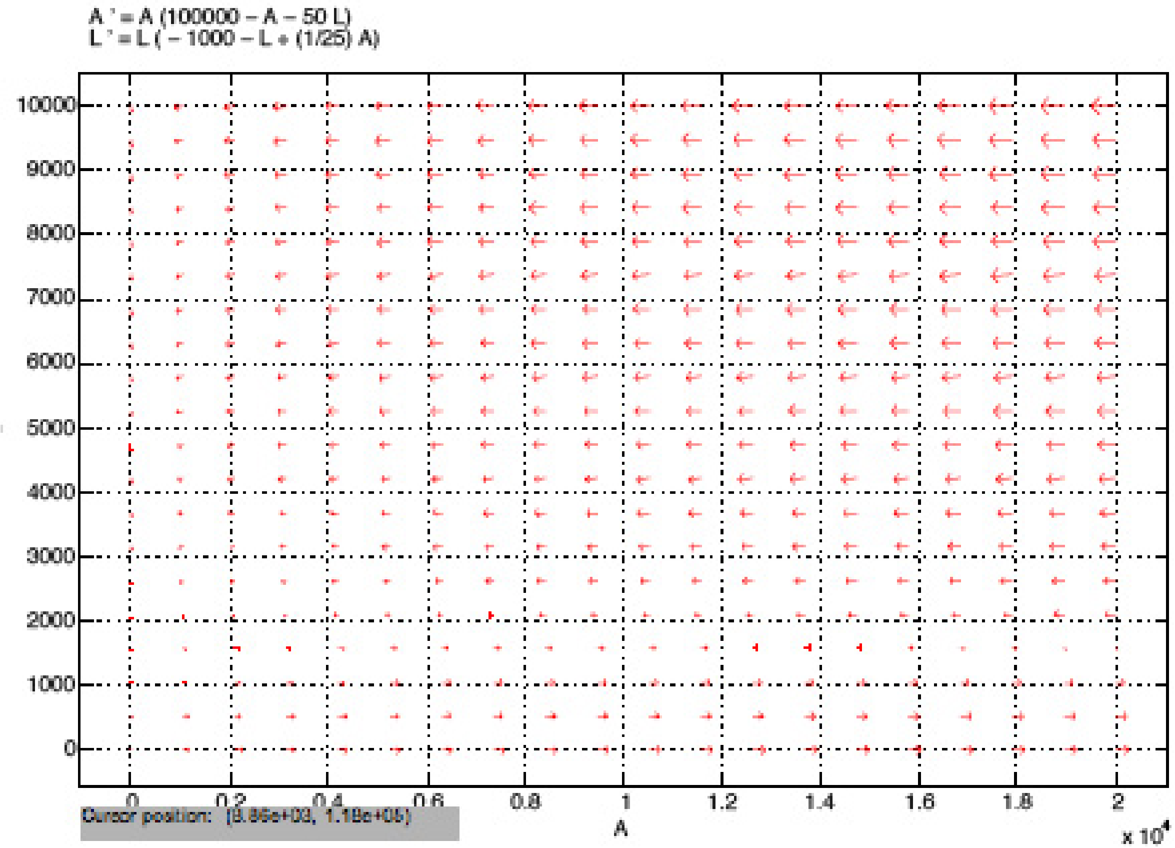



**Part C**

Now together they satisfy

$$A' = A(100000 - A - 50L) \qquad L' = -L(1000 + L - \frac{1}{25}A)$$

The slope field is plotted below. We will have equilibria at the points $(x, y) = (0, 0), (50000, 1000), (100000, 0)$. We will have null clines for $A'$ along $A = 0$ and $L = \frac{100000-A}{50}$. We will have null clines for $L'$ along $L = 0$ and $L = \frac{A-25000}{25}$.




## Problem 18

We are given the equations

$$x' = -.1x - .1x^2 + xy \qquad y' = y - .1y^2 - xy$$

### Part A

This is a predator prey system, because x' increases as y increases and y' decreases. So x is the predator and y is the prey.

### Part B

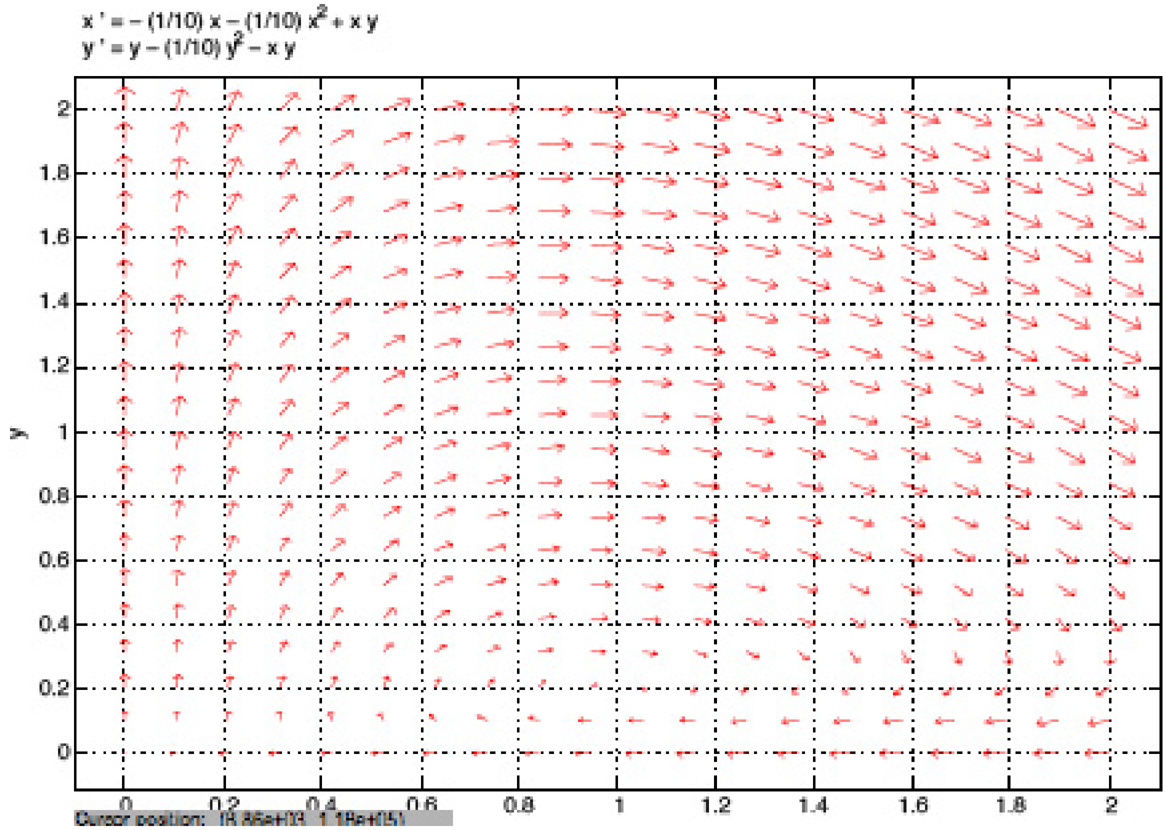
The slope field is plotted below.

### Part C

If $x(0) = 0$ and $y(0) = 1$ we see that $x' = 0$ and $y' = y - .1y^2$. This tells us that $y$ will approach the point where $y' = 0$ or $y = 10$. So as time went on, $x$ would stay 0 and $y$ would asymptotically approach 10. If $x(0)$ and $y(0)$ are both positive then the populations will spiral into the equilibrium point at around $(x, y) = (.9, 1.09)$.

## Part D

Here's the slope field.



# Problem Set 28

Differential Equation Supplement
Page 1041 - Exercises 9, 13, 14

May 9, 2006

## Problem 9

We are given the equations

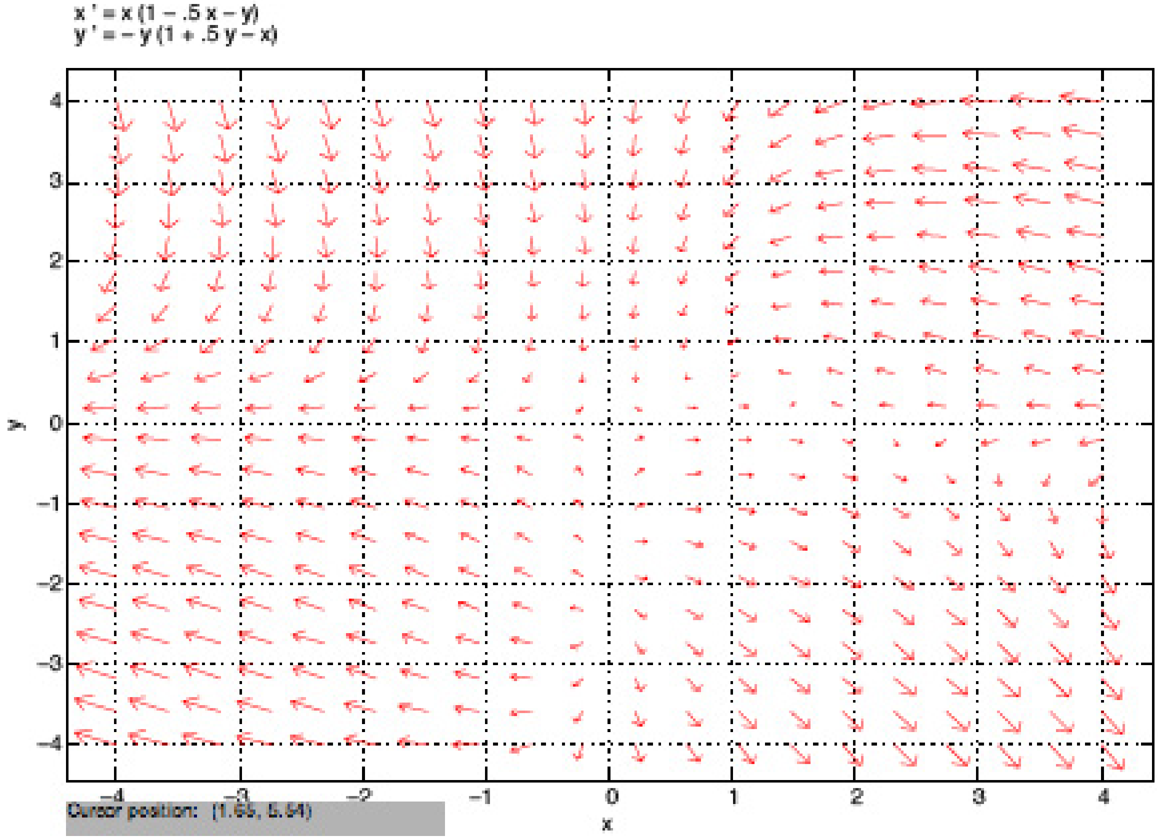
$$x' = x(1 - .5x - y) \qquad y' = -y(1 + .5y - x)$$

### Part A

The equilibria are the points where $x' = y' = 0$. We can see at once that $(0, 0)$ will be a solution. Other solutions will be $(x, y) = (0, -2), (1.2, .4), (2, 0)$.

### Part B

Solving for the null clines for $x' = 0$ give us $x = 0$ and $y = 1 - .5x$. Solving for the null clines for $y' = 0$ gives us $y = 0$ and $y = 2(x - 1)$. The slope field is plotted below.




## Part 13

### Part A

We are given

$$x' = -2 \qquad y' = 4x$$

We can see that $x'$ is always less than zero and the same everywhere, so our arrows must be always pointing somewhere to the left. We can also see that $y' > 0$ for $x > 0$ and $y' < 0$ for $x < 0$. So to the right of the y-axis, our arrows are pointing up and left, and to the left of the y-axis our arrows are pointing down and left. This matches with picture v.

### Part B

We are given

$$x' = 3y \qquad y' = -3x$$

This looks like one from the homework a little bit ago paramterizing a circle, so the choice ix will be what we're looking for. We can see this because above the y-axis, the arrows must be pointing right, below they point left. To the right of the y-axis, they point down and to the left they point up. This we'll describes choice ix, so our first guess was right.

### Part C

We are given that

$$x' = 10x \qquad y' = 10y$$

So to the left of the y-axis, our arrows are pointing left, and to the right they are pointing right. Above the x-axis, they are pointing up, and below the x-axis they are pointing down. This rules it down to vii.

## Problem 14

So in the top left quadrant, when x is negative and y is positive, our arrow is pointing up and to the right. The fact that its pointing right rules out B and D, since they would point left there. The fact that it is pointing up rules out A because for negative x and positive y that must be pointing down. This leaves option C, which is our answer.

**2.** $\left(4 - \frac{1}{2}i\right) - \left(9 + \frac{5}{2}i\right) = (4-9) + \left(-\frac{1}{2} - \frac{5}{2}\right)i = -5 + (-3)i = -5 - 3i$

**4.** $(1-2i)(8-3i) = 8 - 3i - 16i + 6(-1) = 2 - 19i$

**8.** $\dfrac{3+2i}{1-4i} = \dfrac{3+2i}{1-4i} \cdot \dfrac{1+4i}{1+4i} = \dfrac{3+12i+2i+8(-1)}{1^2+4^2} = \dfrac{-5+14i}{17} = -\dfrac{5}{17} + \dfrac{14}{17}i$

**22.** $2x^2 - 2x + 1 = 0 \quad \Leftrightarrow \quad x = \dfrac{-(-2) \pm \sqrt{(-2)^2 - 4(2)(1)}}{2(2)} = \dfrac{2 \pm \sqrt{-4}}{4} = \dfrac{2 \pm 2i}{4} = \dfrac{1}{2} \pm \dfrac{1}{2}i$

**24.** $z^2 + \frac{1}{2}z + \frac{1}{4} = 0 \quad \Leftrightarrow \quad 4z^2 + 2z + 1 = 0 \quad \Leftrightarrow$

$$z = \frac{-2 \pm \sqrt{2^2 - 4(4)(1)}}{2(4)} = \frac{-2 \pm \sqrt{-12}}{8} = \frac{-2 \pm 2\sqrt{3}\,i}{8} = -\frac{1}{4} \pm \frac{\sqrt{3}}{4}i$$

**43.** Using Euler's formula (6) with $y = \dfrac{\pi}{3}$, we have $e^{i\pi/3} = \cos\dfrac{\pi}{3} + i\sin\dfrac{\pi}{3} = \dfrac{1}{2} + \dfrac{\sqrt{3}}{2}i$.

**46.** Using Equation 7 with $x = \pi$ and $y = 1$, we have $e^{\pi+i} = e^{\pi} \cdot e^{1i} = e^{\pi}(\cos 1 + i\sin 1) = e^{\pi}\cos 1 + (e^{\pi}\sin 1)i$.

**48.** Using Formula 6,

$$\begin{aligned} e^{ix} + e^{-ix} &= (\cos x + i\sin x) + [\cos(-x) + i\sin(-x)] \\ &= \cos x + i\sin x + \cos x - i\sin x = 2\cos x \end{aligned}$$

Thus, $\cos x = \dfrac{e^{ix} + e^{-ix}}{2}$.

Similarly,

$$\begin{aligned} e^{ix} - e^{-ix} &= (\cos x + i\sin x) - [\cos(-x) + i\sin(-x)] \\ &= \cos x + i\sin x - \cos x - (-i\sin x) = 2i\sin x \end{aligned}$$

Therefore, $\sin x = \dfrac{e^{ix} - e^{-ix}}{2i}$.