Problem Set 29

Differential Equation Handout
Exercises 21, 22, 23, 24, 25

May 8, 2006

Problem 21

Part A

We are given

$$y'' + 6y' = 7y$$

We'll guess a solution of the form e^{kx} subbing this in and cancelling the exponentials we get that

$$k^2 + 6k - 7 = 0 \Rightarrow (k+7)(k-1) = 0$$

This means that $y(t) = C_1 e^{-7x} + C_2 e^x$

Part B

We are given

$$y'' + 6y' + 9y = 0$$

Guessing a solution of the form e^{kx} and subbing in gives us the relation

$$k^2 + 6k + 9 = 0 \Rightarrow (x+3)^2 = 0$$

Since we get a double root, our solution is

$$C_1 e^{-3x} + C_2 x e^{-3x}$$

Part C

We are given that

$$y'' + 5y' + 6y = 0$$

Again, guessing a solution of the form e^{kx} gives us the relation

$$k^2 + 5k + 6 = 0 \Rightarrow (k+3)(k+2) = 0$$

Thus our solution is just

$$C_1 e^{-3x} + C_2 e^{-2x}$$

Problem 22

Part A

We found that $y = C_1 e^{-7x} + C_2 e^{x}$. If $y(0) = -2$ then we have the relation that $C_1 + C_2 = -2$. The relation $y'(0) = 0$ gives us the relation that $-7C_1 + C_2 = 0$. Putting these together we yield the solution

$$y = -\frac{1}{4}e^{-7x} - \frac{7}{4}e^{x}$$

As $x \to \infty$ the first term goes to 0, but the second term goes to negative infinity.

Part B

We found that $y = C_1 e^{-3x} + C_2 x e^{-3x}$. The boundary condition of $y(0) = -2$ gives us $C_1 = -2$. The boundary condition of $y'(0) = 0$ will then give us that $C_2 = -6$. Thus our final solution is just

$$y = -2e^{-3x} - 6xe^{-3x}$$

As $x \to \infty$ both the exponentials die, even faster than the x term, so the limit will be 0.

Part C

We found that $y = C_1 e^{-3x} + C_2 e^{-2x}$. Plugging in the two boundary conditions like before we just get

$$y = 4e^{-3x} - 6e^{-2x}$$

This will approach 0 as $x \to \infty$ since both the exponentials die.

Problem 23

Part A

For this we need to solve the given differential equation

$$x'' + 4x' + 3x = 0$$

Guessing a solution of e^{kt} we get the relation

$$k^2 + 4k + 3 = 0 \Rightarrow (k+3)(k+1) = 0$$

Thus our solution will be of the form $C_1 e^{-3t} + C_2 e^{-t}$. Our initial condition as given above are $x(0) = 1$ and $x'(0) = 2$. Plugging these in gives us the final solution of

$$x(t) = -\frac{3}{2}e^{-3t} + \frac{5}{2}e^{-t}$$

Part B

Let's see if we can find a time such that $x(t) = 0$.

$$0 = -\frac{3}{2}e^{-3t} + \frac{5}{2}e^{-t} \Rightarrow \frac{5}{3} = e^{-2t} \Rightarrow t = -\frac{1}{2}\ln\left(\frac{5}{3}\right)$$

This gives a solution of $t < 0$ so it never does cross the point $x = 0$.

Part C

So we want to find the max. For this we take the derivative and set it equal to 0. Taking the derivative yields

$$x' = \frac{9}{2}e^{-3t} - \frac{5}{2}e^{-t} = 0$$

Solving for t yields $t = \frac{1}{2}\ln\left(\frac{9}{5}\right)$

Problem 24

Part A

Solving for $x(t) = 0$ we get that

$$\frac{-C_1}{C_2} = e^{(b-a)t} \Rightarrow t = \frac{1}{b-a}\ln\frac{-C_1}{C_2}$$

This gives us a unique solution given the four constants. We can also think that if it must cross zero at least once, then C_1 and C_2 must be of opposite sign. Now, for it to cross zero more than once, we must have a max or min in between which means the derivative is 0, which means that a and b must be of the same sign. If this is the case the our function will either be monotonically increasing or decreasing after the first zero. Also, its good to note that we're excluding the case where $C_1 = -C_2$ and $a = b$, in which case its zero everywhere.

Part B

Solving for 0 we get

$$0 = C_1 + C_2 t \Rightarrow t = \frac{-C_1}{C_2}$$

To show its the only zero we can group the terms in the fashion

$$e^{at}\left(C_1 + C_2 t\right)$$

The first term is non-zero for all t, which means the second term is nonzero. This is linear and will only have a single 0.

Part C

If the characteristic equation has one or two real roots, then we will get solutions of the form given in parts a and b. As shown above, each of these only cross the equilibrium once, and thus cannot model a mass that oscillates back and forth.

Part 25

We know that

$$e^x = \sum_0^\infty \frac{x^k}{k!}$$

Looking at the taylor expansions for sine and cosine, we see that

$$e^{ix} = \cos x + i \sin x$$

So this tells us that

$$e^{bit} = \cos bt + i \sin bt$$

and we get that $e^{i\pi} = -1$.