1. (a) See Definition 8.1.1.

(b) See Definition 8.2.2.

(c) The terms of the sequence $\{a_n\}$ approach 3 as n becomes large.

(d) By adding sufficiently many terms of the series, we can make the partial sums as close to 3 as we like.

2. (a) See the definition on page 563.

(b) A sequence is monotonic if it is either increasing or decreasing.

(c) By Theorem 8.1.7, every bounded, monotonic sequence is convergent.

3. (a) See (4) in Section 8.2.

(b) The p-series $\sum\limits_{n=1}^{\infty} \dfrac{1}{n^p}$ is convergent if $p > 1$.

4. If $\sum a_n = 3$, then $\lim\limits_{n \to \infty} a_n = 0$ and $\lim\limits_{n \to \infty} s_n = 3$.

5. (a) See the Test for Divergence on page 572.

(b) See the Integral Test on page 578.

(c) See the Comparison Test on page 580.

(d) See the Limit Comparison Test on page 582.

(e) See the Alternating Series Test on page 587.

(f) See the Ratio Test on page 591.

6. (a) A series $\sum a_n$ is called *absolutely convergent* if the series of absolute values $\sum |a_n|$ is convergent.

(b) If a series $\sum a_n$ is absolutely convergent, then it is convergent.

7. (a) Use (4) in Section 8.3.

(b) See Example 8 in Section 8.3.

(c) By adding terms until you reach the desired accuracy given by the Alternating Series Estimation Theorem on page 588.

8. (a) $\sum_{n=0}^{\infty} c_n (x - a)^n$

(b) Given the power series $\sum_{n=0}^{\infty} c_n (x - a)^n$, the radius of convergence is:

(i) 0 if the series converges only when $x = a$

(ii) ∞ if the series converges for all x, or

(iii) a positive number R such that the series converges if $|x - a| < R$ and diverges if $|x - a| > R$.

(c) The interval of convergence of a power series is the interval that consists of all values of x for which the series converges. Corresponding to the cases in part (b), the interval of convergence is: (i) the single point $\{a\}$, (ii) all real numbers, that is, the real number line $(-\infty, \infty)$, or (iii) an interval with endpoints $a - R$ and $a + R$ which can contain neither, either, or both of the endpoints. In this case, we must test the series for convergence at each endpoint to determine the interval of convergence.

9. (a), (b) See Theorem 8.6.2.

10. (a) $T_n(x) = \sum_{i=0}^{n} \dfrac{f^{(i)}(a)}{i!} (x - a)^i$

(b) $\sum_{n=0}^{\infty} \dfrac{f^{(n)}(a)}{n!} (x - a)^n$

(c) $\sum_{n=0}^{\infty} \dfrac{f^{(n)}(0)}{n!} x^n$ $[a = 0$ in part (b)$]$

(d) See Theorem 8.7.8.

(e) See Taylor's Inequality (8.7.9).

11. (a) – (e) See the table on page 612.

12. See the Binomial Series (8.8.2) for the expansion. The radius of convergence for the binomial series is 1.

1. False.   See Note 2 after Theorem 8.2.6.

2. False.   The series $\sum\limits_{n=1}^{\infty} n^{-\sin 1} = \sum\limits_{n=1}^{\infty} \dfrac{1}{n^{\sin 1}}$ is a p-series with $p = \sin 1 \approx 0.84 \leq 1$, so the series diverges.

3. True.   If $\lim\limits_{n\to\infty} a_n = L$, then given any $\varepsilon > 0$, we can find a positive integer N such that $|a_n - L| < \varepsilon$ whenever
$n > N$. If $n > N$, then $2n + 1 > N$ and $|a_{2n+1} - L| < \varepsilon$. Thus, $\lim\limits_{n\to\infty} a_{2n+1} = L$.

4. True by Theorem 8.5.3.
   Or: Use the Comparison Test to show that $\sum c_n(-2)^n$ converges absolutely.

5. False.   For example, take $c_n = (-1)^n/(n6^n)$.

6. True by Theorem 8.5.3.

7. False, since $\lim\limits_{n\to\infty} \left| \dfrac{a_{n+1}}{a_n} \right| = \lim\limits_{n\to\infty} \left| \dfrac{1}{(n+1)^3} \cdot \dfrac{n^3}{1} \right| = \lim\limits_{n\to\infty} \left| \dfrac{n^3}{(n+1)^3} \cdot \dfrac{1/n^3}{1/n^3} \right| = \lim\limits_{n\to\infty} \dfrac{1}{(1+1/n)^3} = 1.$

8. True, since $\lim\limits_{n\to\infty} \left| \dfrac{a_{n+1}}{a_n} \right| = \lim\limits_{n\to\infty} \left| \dfrac{1}{(n+1)!} \cdot \dfrac{n!}{1} \right| = \lim\limits_{n\to\infty} \dfrac{1}{n+1} = 0 < 1.$

9. False.   See the note after Example 4 in Section 8.3.

10. True, since $\dfrac{1}{e} = e^{-1}$ and $e^x = \sum\limits_{n=0}^{\infty} \dfrac{x^n}{n!}$, so $e^{-1} = \sum\limits_{n=0}^{\infty} \dfrac{(-1)^n}{n!}$.

11. True.   See (6) in Section 8.1.

12. True, because if $\sum |a_n|$ is convergent, then so is $\sum a_n$ by Theorem 8.4.1.

13. True.   By Theorem 8.7.5 the coefficient of x^3 is $\dfrac{f'''(0)}{3!} = \dfrac{1}{3} \;\Rightarrow\; f'''(0) = 2$.

                                 Or: Use Theorem 8.6.2 to differentiate f three times.

14. False.   Let $a_n = n$ and $b_n = -n$. Then $\{a_n\}$ and $\{b_n\}$ are divergent, but $a_n + b_n = 0$, so $\{a_n + b_n\}$ is convergent.

15. False.   For example, let $a_n = b_n = (-1)^n$. Then $\{a_n\}$ and $\{b_n\}$ are divergent, but $a_n b_n = 1$, so $\{a_n b_n\}$ is convergent.

16. True by the Monotonic Sequence Theorem, since $\{a_n\}$ is decreasing and $0 < a_n \le a_1$ for all n $\;\Rightarrow\;$ $\{a_n\}$ is bounded.

17. True by Theorem 8.4.1. $\left[\sum (-1)^n \, a_n \text{ is absolutely convergent and hence convergent.}\right]$

18. True. $\displaystyle\lim_{n\to\infty} \frac{a_{n+1}}{a_n} < 1 \;\Rightarrow\; \sum a_n$ converges (Ratio Test) $\;\Rightarrow\; \displaystyle\lim_{n\to\infty} a_n = 0$ [Theorem 8.2.6].

1. $\left\{\dfrac{2+n^3}{1+2n^3}\right\}$ converges since $\displaystyle\lim_{n\to\infty}\dfrac{2+n^3}{1+2n^3}=\lim_{n\to\infty}\dfrac{2/n^3+1}{1/n^3+2}=\dfrac{1}{2}$.

2. $a_n=\dfrac{9^{n+1}}{10^n}=9\cdot\left(\tfrac{9}{10}\right)^n$, so $\displaystyle\lim_{n\to\infty}a_n=9\lim_{n\to\infty}\left(\tfrac{9}{10}\right)^n=9\cdot 0=0$ by (8.1.6).

3. $\displaystyle\lim_{n\to\infty}a_n=\lim_{n\to\infty}\dfrac{n^3}{1+n^2}=\lim_{n\to\infty}\dfrac{n}{1/n^2+1}=\infty$, so the sequence diverges.

4. $a_n=\cos(n\pi/2)$, so $a_n=0$ if n is odd and $a_n=\pm1$ if n is even. As n increases, a_n keeps cycling through the values $0, 1,$ $0, -1$, so the sequence $\{a_n\}$ is divergent.

5. $|a_n|=\left|\dfrac{n\sin n}{n^2+1}\right|\le\dfrac{n}{n^2+1}<\dfrac{1}{n}$, so $|a_n|\to 0$ as $n\to\infty$. Thus, $\displaystyle\lim_{n\to\infty}a_n=0$. The sequence $\{a_n\}$ is convergent.

6. $a_n=\dfrac{\ln n}{\sqrt{n}}$. Let $f(x)=\dfrac{\ln x}{\sqrt{x}}$ for $x>0$. Then $\displaystyle\lim_{x\to\infty}f(x)=\lim_{x\to\infty}\dfrac{\ln x}{\sqrt{x}}\overset{\text{H}}{=}\lim_{x\to\infty}\dfrac{1/x}{1/(2\sqrt{x})}=\lim_{x\to\infty}\dfrac{2}{\sqrt{x}}=0$. Thus, by

Theorem 2 in Section 8.1, $\{a_n\}$ converges and $\displaystyle\lim_{n\to\infty}a_n=0$.

7. $\left\{\left(1+\dfrac{3}{n}\right)^{4n}\right\}$ is convergent. Let $y=\left(1+\dfrac{3}{x}\right)^{4x}$. Then

$$\lim_{x\to\infty}\ln y=\lim_{x\to\infty}4x\ln(1+3/x)=\lim_{x\to\infty}\dfrac{\ln(1+3/x)}{1/(4x)}\overset{\text{H}}{=}\lim_{x\to\infty}\dfrac{\dfrac{1}{1+3/x}\left(-\dfrac{3}{x^2}\right)}{-1/(4x^2)}=\lim_{x\to\infty}\dfrac{12}{1+3/x}=12$$

so $\displaystyle\lim_{x\to\infty}y=\lim_{n\to\infty}\left(1+\dfrac{3}{n}\right)^{4n}=e^{12}$.

Or: Use Exercise 4.5.38.

8. We use induction, hypothesizing that $a_{n-1}<a_n<2$. Note first that $1<a_2=\tfrac{1}{3}(1+4)=\tfrac{5}{3}<2$, so the hypothesis holds for $n=2$. Now assume that $a_{k-1}<a_k<2$. Then $a_k=\tfrac{1}{3}(a_{k-1}+4)<\tfrac{1}{3}(a_k+4)<\tfrac{1}{3}(2+4)=2$. So $a_k<a_{k+1}<2$, and the induction is complete. To find the limit of the sequence, we note that $L=\displaystyle\lim_{n\to\infty}a_n=\lim_{n\to\infty}a_{n+1}\ \Rightarrow$

$L=\tfrac{1}{3}(L+4)\ \Rightarrow\ L=2$.

9. $\dfrac{n}{n^3+1} < \dfrac{n}{n^3} = \dfrac{1}{n^2}$, so $\displaystyle\sum_{n=1}^{\infty} \dfrac{n}{n^3+1}$ converges by the Comparison Test with the convergent p-series $\displaystyle\sum_{n=1}^{\infty} \dfrac{1}{n^2}$ $(p=2>1)$.

10. Let $a_n = \dfrac{n^2+1}{n^3+1}$ and $b_n = \dfrac{1}{n}$, so $\displaystyle\lim_{n\to\infty} \dfrac{a_n}{b_n} = \lim_{n\to\infty} \dfrac{n^3+n}{n^3+1} = \lim_{n\to\infty} \dfrac{1+1/n^2}{1+1/n^3} = 1 > 0$. Since $\sum_{n=1}^{\infty} b_n$ is the divergent

harmonic series, $\sum_{n=1}^{\infty} a_n$ also diverges by the Limit Comparison Test.

11. $\displaystyle\lim_{n\to\infty} \left| \dfrac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left[\dfrac{(n+1)^3}{5^{n+1}} \cdot \dfrac{5^n}{n^3} \right] = \lim_{n\to\infty} \left(1 + \dfrac{1}{n} \right)^3 \cdot \dfrac{1}{5} = \dfrac{1}{5} < 1$, so $\displaystyle\sum_{n=1}^{\infty} \dfrac{n^3}{5^n}$ converges by the Ratio Test.

12. Let $b_n = \dfrac{1}{\sqrt{n+1}}$. Then b_n is positive for $n \geq 1$, the sequence $\{b_n\}$ is decreasing, and $\displaystyle\lim_{n\to\infty} b_n = 0$, so the series

$\displaystyle\sum_{n=1}^{\infty} \dfrac{(-1)^n}{\sqrt{n+1}}$ converges by the Alternating Series Test.

13. Let $f(x) = \dfrac{1}{x\sqrt{\ln x}}$. Then f is continuous, positive, and decreasing on $[2, \infty)$, so the Integral Test applies.

$$\int_2^{\infty} f(x)\,dx = \lim_{t\to\infty} \int_2^t \dfrac{1}{x\sqrt{\ln x}}\,dx \quad \begin{bmatrix} u = \ln x, \\ du = \dfrac{1}{x}\,dx \end{bmatrix} = \lim_{t\to\infty} \int_{\ln 2}^{\ln t} u^{-1/2}\,du$$

$$= \lim_{t\to\infty} [2\sqrt{u}]_{\ln 2}^{\ln t} = \lim_{t\to\infty} \left(2\sqrt{\ln t} - 2\sqrt{\ln 2} \right) = \infty, \text{ so the series } \sum_{n=2}^{\infty} \dfrac{1}{n\sqrt{\ln n}} \text{ diverges.}$$

14. $\displaystyle\lim_{n\to\infty} \dfrac{n}{3n+1} = \dfrac{1}{3}$, so $\displaystyle\lim_{n\to\infty} \ln\left(\dfrac{n}{3n+1} \right) = \ln\tfrac{1}{3} \neq 0$. Thus, the series $\displaystyle\sum_{n=1}^{\infty} \ln\left(\dfrac{n}{3n+1} \right)$ diverges by the Test for

Divergence.

15. $b_n = \dfrac{\sqrt{n}}{n+1} > 0$, $\{b_n\}$ is decreasing, and $\displaystyle\lim_{n\to\infty} b_n = 0$, so the series $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \dfrac{\sqrt{n}}{n+1}$ converges by the Alternating Series

Test.

16. $|a_n| = \left| \dfrac{\cos 3n}{1+(1.2)^n} \right| \leq \dfrac{1}{1+(1.2)^n} < \dfrac{1}{(1.2)^n} = \left(\dfrac{5}{6} \right)^n$, so $\displaystyle\sum_{n=1}^{\infty} |a_n|$ converges by comparison with the convergent geometric

series $\displaystyle\sum_{n=1}^{\infty} \left(\dfrac{5}{6} \right)^n$ $\left(r = \tfrac{5}{6} < 1 \right)$. It follows that $\displaystyle\sum_{n=1}^{\infty} a_n$ converges (by Theorem 1 in Section 8.4).

17. $\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{1\cdot 3\cdot 5\cdots\cdots(2n-1)(2n+1)}{5^{n+1}\,(n+1)!}\cdot\frac{5^n n!}{1\cdot 3\cdot 5\cdots\cdots(2n-1)} = \lim_{n\to\infty}\frac{2n+1}{5(n+1)} = \frac{2}{5} < 1$, so the series converges by the Ratio Test.

18. $\displaystyle\sum_{n=1}^{\infty}\frac{(-5)^{2n}}{n^2\,9^n} = \sum_{n=1}^{\infty}\frac{1}{n^2}\left(\frac{25}{9}\right)^n$. Now $\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{25^{n+1}}{(n+1)^2\cdot 9^{n+1}}\cdot\frac{n^2\cdot 9^n}{25^n} = \lim_{n\to\infty}\frac{25n^2}{9(n+1)^2} = \frac{25}{9} > 1$, so the series diverges by the Ratio Test.

19. $\displaystyle\frac{2^{2n+1}}{5^n} = \frac{2^{2n}\cdot 2^1}{5^n} = \frac{(2^2)^n\cdot 2}{5^n} = 2\left(\frac{4}{5}\right)^n$, so $\displaystyle\sum_{n=1}^{\infty}\frac{2^{2n+1}}{5^n} = 2\sum_{n=1}^{\infty}\left(\frac{4}{5}\right)^n$ is a geometric series with $a = \frac{8}{5}$ and $r = \frac{4}{5}$. Since $|r| = \frac{4}{5} < 1$, the series converges to $\displaystyle\frac{a}{1-r} = \frac{8/5}{1-4/5} = \frac{8/5}{1/5} = 8.$

20. $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n\pi^n}{3^{2n}(2n)!} = \sum_{n=0}^{\infty}(-1)^n\frac{1}{(2n)!}\cdot\frac{\pi^n}{3^{2n}} = \sum_{n=0}^{\infty}(-1)^n\frac{1}{(2n)!}\cdot\left(\frac{\sqrt{\pi}}{3}\right)^{2n} = \cos\left(\frac{\sqrt{\pi}}{3}\right)$ since $\displaystyle\cos x = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}$ for all x.

21. $\displaystyle\sum_{n=1}^{\infty}\left[\tan^{-1}(n+1) - \tan^{-1} n\right] = \lim_{n\to\infty}s_n$
$$= \lim_{n\to\infty}\left[(\tan^{-1}2 - \tan^{-1}1) + (\tan^{-1}3 - \tan^{-1}2) + \cdots\right.$$
$$\left. + (\tan^{-1}(n+1) - \tan^{-1} n)\right]$$
$$= \lim_{n\to\infty}\left[\tan^{-1}(n+1) - \tan^{-1}1\right] = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$$

22. $\displaystyle 1 - e + \frac{e^2}{2!} - \frac{e^3}{3!} + \frac{e^4}{4!} - \cdots = \sum_{n=0}^{\infty}(-1)^n\frac{e^n}{n!} = \sum_{n=0}^{\infty}\frac{(-e)^n}{n!} = e^{-e}$ since $\displaystyle e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!}$ for all x.

23. $\displaystyle 1.2345345345\ldots = 1.2 + 0.0\overline{345} = \frac{12}{10} + \frac{345/10{,}000}{1 - 1/1000} = \frac{12}{10} + \frac{345}{9990} = \frac{4111}{3330}$

24. $\displaystyle\sum_{n=1}^{\infty}(\ln x)^n$ is a geometric series which converges whenever $|\ln x| < 1 \;\Rightarrow\; -1 < \ln x < 1 \;\Rightarrow\; e^{-1} < x < e.$

25. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^5} = 1 - \frac{1}{32} + \frac{1}{243} - \frac{1}{1024} + \frac{1}{3125} - \frac{1}{7776} + \frac{1}{16,807} - \frac{1}{32,768} + \cdots.$

Since $b_8 = \dfrac{1}{8^5} = \dfrac{1}{32,768} < 0.000031,$ $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^5} \approx \sum_{n=1}^{7} \frac{(-1)^{n+1}}{n^5} \approx 0.9721.$

26. (a) $s_5 = \displaystyle\sum_{n=1}^{5} \frac{1}{n^6} = 1 + \frac{1}{2^6} + \cdots + \frac{1}{5^6} \approx 1.017305.$ The series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^6}$ converges by the Integral Test, so we estimate the

remainder R_5 with (8.3.3): $R_5 \le \displaystyle\int_{5}^{\infty} \frac{dx}{x^6} = \left[-\frac{x^{-5}}{5} \right]_{5}^{\infty} = \frac{5^{-5}}{5} = 0.000064.$ So the error is at most $0.000064.$

(b) In general, $R_n \le \displaystyle\int_{n}^{\infty} \frac{dx}{x^6} = \frac{1}{5n^5}.$ If we take $n = 9,$ then $s_9 \approx 1.01734$ and $R_9 \le \dfrac{1}{5 \cdot 9^5} \approx 3.4 \times 10^{-6}.$ So to five

decimal places, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^5} \approx \sum_{n=1}^{9} \frac{1}{n^5} \approx 1.01734.$

Another method: Use (8.3.4) instead of (8.3.3).

27. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2 + 5^n} \approx \sum_{n=1}^{8} \frac{1}{2 + 5^n} \approx 0.18976224.$ To estimate the error, note that $\dfrac{1}{2 + 5^n} < \dfrac{1}{5^n},$ so the remainder term is

$R_8 = \displaystyle\sum_{n=9}^{\infty} \frac{1}{2 + 5^n} < \sum_{n=9}^{\infty} \frac{1}{5^n} = \frac{1/5^9}{1 - 1/5} = 6.4 \times 10^{-7}$ $\left[\text{geometric series with } a = \frac{1}{5^9} \text{ and } r = \frac{1}{5}\right].$

28. (a) $\displaystyle\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(n+1)^{n+1}}{[2(n+1)]!} \cdot \frac{(2n)!}{n^n} \right| = \lim_{n\to\infty} \frac{(n+1)^n (n+1)^1}{(2n+2)(2n+1)n^n} = \lim_{n\to\infty} \left(\frac{n+1}{n} \right)^n \frac{1}{2(2n+1)}$

$= \displaystyle\lim_{n\to\infty} \left(1 + \frac{1}{n} \right)^n \frac{1}{2(2n+1)} = e \cdot 0 = 0 < 1$

so the series converges by the Ratio Test.

(b) The series in part (a) is convergent, so $\displaystyle\lim_{n\to\infty} a_n = 0$ by Theorem 8.2.6.

29. Use the Limit Comparison Test. $\displaystyle\lim_{n\to\infty} \left| \frac{\left(\frac{n+1}{n}\right)a_n}{a_n} \right| = \lim_{n\to\infty} \frac{n+1}{n} = \lim_{n\to\infty} \left(1 + \frac{1}{n} \right) = 1 > 0.$

Since $\sum |a_n|$ is convergent, so is $\displaystyle\sum \left| \left(\frac{n+1}{n} \right) a_n \right|,$ by the Limit Comparison Test.

30. $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty}\left|\dfrac{x^{n+1}}{(n+1)^2\,5^{n+1}}\cdot\dfrac{n^2 5^n}{x^n}\right| = \lim\limits_{n\to\infty}\dfrac{1}{(1+1/n)^2}\dfrac{|x|}{5} = \dfrac{|x|}{5}$, so by the Ratio Test, $\sum\limits_{n=1}^{\infty}(-1)^n\dfrac{x^n}{n^2\,5^n}$

converges when $\dfrac{|x|}{5}<1 \quad\Leftrightarrow\quad |x|<5$, so $R=5$. When $x=-5$, the series becomes the convergent p-series $\sum\limits_{n=1}^{\infty}\dfrac{1}{n^2}$ with

$p=2>1$. When $x=5$, the series becomes $\sum\limits_{n=1}^{\infty}\dfrac{(-1)^n}{n^2}$, which converges by the Alternating Series Test. Thus, $I=[-5,5]$.

31. $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty}\left[\dfrac{|x+2|^{n+1}}{(n+1)\,4^{n+1}}\cdot\dfrac{n\,4^n}{|x+2|^n}\right] = \lim\limits_{n\to\infty}\left[\dfrac{n}{n+1}\dfrac{|x+2|}{4}\right] = \dfrac{|x+2|}{4}<1 \quad\Leftrightarrow\quad |x+2|<4$, so $R=4$.

$|x+2|<4 \quad\Leftrightarrow\quad -4<x+2<4 \quad\Leftrightarrow\quad -6<x<2$. If $x=-6$, then the series $\sum\limits_{n=1}^{\infty}\dfrac{(x+2)^n}{n4^n}$ becomes

$\sum\limits_{n=1}^{\infty}\dfrac{(-4)^n}{n4^n} = \sum\limits_{n=1}^{\infty}\dfrac{(-1)^n}{n}$, the alternating harmonic series, which converges by the Alternating Series Test. When $x=2$, the

series becomes the harmonic series $\sum\limits_{n=1}^{\infty}\dfrac{1}{n}$, which diverges. Thus, $I=[-6,2)$.

32. $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty}\left|\dfrac{2^{n+1}(x-2)^{n+1}}{(n+3)!}\cdot\dfrac{(n+2)!}{2^n(x-2)^n}\right| = \lim\limits_{n\to\infty}\dfrac{2}{n+3}|x-2| = 0<1$, so the series $\sum\limits_{n=1}^{\infty}\dfrac{2^n(x-2)^n}{(n+2)!}$

converges for all x. $R=\infty$ and $I=(-\infty,\infty)$.

33. $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty}\left|\dfrac{2^{n+1}(x-3)^{n+1}}{\sqrt{n+4}}\cdot\dfrac{\sqrt{n+3}}{2^n(x-3)^n}\right| = 2|x-3|\lim\limits_{n\to\infty}\sqrt{\dfrac{n+3}{n+4}} = 2|x-3|<1 \quad\Leftrightarrow\quad |x-3|<\tfrac12$, so

$R=\tfrac12$. $|x-3|<\tfrac12 \quad\Leftrightarrow\quad -\tfrac12<x-3<\tfrac12 \quad\Leftrightarrow\quad \tfrac52<x<\tfrac72$. For $x=\tfrac72$, the series $\sum\limits_{n=1}^{\infty}\dfrac{2^n(x-3)^n}{\sqrt{n+3}}$ becomes

$\sum\limits_{n=0}^{\infty}\dfrac{1}{\sqrt{n+3}} = \sum\limits_{n=3}^{\infty}\dfrac{1}{n^{1/2}}$, which diverges ($p=\tfrac12\le 1$), but for $x=\tfrac52$, we get $\sum\limits_{n=0}^{\infty}\dfrac{(-1)^n}{\sqrt{n+3}}$, which is a convergent

alternating series, so $I=\left[\tfrac52,\tfrac72\right)$.

34. $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty}\left|\dfrac{(2n+2)!\,x^{n+1}}{[(n+1)!]^2}\cdot\dfrac{(n!)^2}{(2n)!\,x^n}\right| = \lim\limits_{n\to\infty}\dfrac{(2n+2)(2n+1)}{(n+1)(n+1)}|x| = 4\,|x|$.

35.

n	$f^{(n)}(x)$	$f^{(n)}\left(\frac{\pi}{6}\right)$
0	$\sin x$	$\frac{1}{2}$
1	$\cos x$	$\frac{\sqrt{3}}{2}$
2	$-\sin x$	$-\frac{1}{2}$
3	$-\cos x$	$-\frac{\sqrt{3}}{2}$
4	$\sin x$	$\frac{1}{2}$
$\vdots$	$\vdots$	$\vdots$

$$\sin x = f\left(\frac{\pi}{6}\right) + f'\left(\frac{\pi}{6}\right)\left(x - \frac{\pi}{6}\right) + \frac{f''\left(\frac{\pi}{6}\right)}{2!}\left(x - \frac{\pi}{6}\right)^2 + \frac{f^{(3)}\left(\frac{\pi}{6}\right)}{3!}\left(x - \frac{\pi}{6}\right)^3 + \frac{f^{(4)}\left(\frac{\pi}{6}\right)}{4!}\left(x - \frac{\pi}{6}\right)^4 + \cdots$$

$$= \frac{1}{2}\left[1 - \frac{1}{2!}\left(x - \frac{\pi}{6}\right)^2 + \frac{1}{4!}\left(x - \frac{\pi}{6}\right)^4 - \cdots\right] + \frac{\sqrt{3}}{2}\left[\left(x - \frac{\pi}{6}\right) - \frac{1}{3!}\left(x - \frac{\pi}{6}\right)^3 + \cdots\right]$$

$$= \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n\frac{1}{(2n)!}\left(x - \frac{\pi}{6}\right)^{2n} + \frac{\sqrt{3}}{2}\sum_{n=0}^{\infty}(-1)^n\frac{1}{(2n+1)!}\left(x - \frac{\pi}{6}\right)^{2n+1}$$

36.

n	$f^{(n)}(x)$	$f^{(n)}\left(\frac{\pi}{3}\right)$
0	$\cos x$	$\frac{1}{2}$
1	$-\sin x$	$-\frac{\sqrt{3}}{2}$
2	$-\cos x$	$-\frac{1}{2}$
3	$\sin x$	$\frac{\sqrt{3}}{2}$
4	$\cos x$	$\frac{1}{2}$
$\vdots$	$\vdots$	$\vdots$

$$\cos x = f\left(\frac{\pi}{3}\right) + f'\left(\frac{\pi}{3}\right)\left(x - \frac{\pi}{3}\right) + \frac{f''\left(\frac{\pi}{3}\right)}{2!}\left(x - \frac{\pi}{3}\right)^2 + \frac{f^{(3)}\left(\frac{\pi}{3}\right)}{3!}\left(x - \frac{\pi}{3}\right)^3 + \frac{f^{(4)}\left(\frac{\pi}{3}\right)}{4!}\left(x - \frac{\pi}{3}\right)^4 + \cdots$$

$$= \frac{1}{2}\left[1 - \frac{1}{2!}\left(x - \frac{\pi}{3}\right)^2 + \frac{1}{4!}\left(x - \frac{\pi}{3}\right)^4 - \cdots\right] + \frac{\sqrt{3}}{2}\left[-\left(x - \frac{\pi}{3}\right) + \frac{1}{3!}\left(x - \frac{\pi}{3}\right)^3 - \cdots\right]$$

$$= \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n\frac{1}{(2n)!}\left(x - \frac{\pi}{3}\right)^{2n} + \frac{\sqrt{3}}{2}\sum_{n=0}^{\infty}(-1)^{n+1}\frac{1}{(2n+1)!}\left(x - \frac{\pi}{3}\right)^{2n+1}$$

37. $\dfrac{1}{1+x} = \dfrac{1}{1-(-x)} = \sum\limits_{n=0}^{\infty} (-x)^n = \sum\limits_{n=0}^{\infty} (-1)^n x^n$ for $|x| < 1$ $\Rightarrow$ $\dfrac{x^2}{1+x} = \sum\limits_{n=0}^{\infty} (-1)^n x^{n+2}$ with $R = 1$.

38. $\tan^{-1} x = \sum\limits_{n=0}^{\infty} (-1)^n \dfrac{x^{2n+1}}{2n+1}$ with interval of convergence $[-1, 1]$, so

$\tan^{-1}(x^2) = \sum\limits_{n=0}^{\infty} (-1)^n \dfrac{(x^2)^{2n+1}}{2n+1} = \sum\limits_{n=0}^{\infty} (-1)^n \dfrac{x^{4n+2}}{2n+1}$, which converges when $x^2 \in [-1, 1]$ $\Leftrightarrow$ $x \in [-1, 1]$.

Therefore, $R = 1$.

39. $\dfrac{1}{1-x} = \sum\limits_{n=0}^{\infty} x^n$ for $|x| < 1$ $\Rightarrow$ $\ln(1-x) = -\int \dfrac{dx}{1-x} = -\int \sum\limits_{n=0}^{\infty} x^n \, dx = C - \sum\limits_{n=0}^{\infty} \dfrac{x^{n+1}}{n+1}$.

$\ln(1-0) = C - 0$ $\Rightarrow$ $C = 0$ $\Rightarrow$ $\ln(1-x) = -\sum\limits_{n=0}^{\infty} \dfrac{x^{n+1}}{n+1} = \sum\limits_{n=1}^{\infty} \dfrac{-x^n}{n}$ with $R = 1$.

40. $e^x = \sum\limits_{n=0}^{\infty} \dfrac{x^n}{n!}$ $\Rightarrow$ $e^{2x} = \sum\limits_{n=0}^{\infty} \dfrac{(2x)^n}{n!}$ $\Rightarrow$ $xe^{2x} = x \sum\limits_{n=0}^{\infty} \dfrac{2^n x^n}{n!} = \sum\limits_{n=0}^{\infty} \dfrac{2^n x^{n+1}}{n!}$, $R = \infty$

41. $\sin x = \sum\limits_{n=0}^{\infty} \dfrac{(-1)^n x^{2n+1}}{(2n+1)!}$ $\Rightarrow$ $\sin(x^4) = \sum\limits_{n=0}^{\infty} \dfrac{(-1)^n (x^4)^{2n+1}}{(2n+1)!} = \sum\limits_{n=0}^{\infty} \dfrac{(-1)^n x^{8n+4}}{(2n+1)!}$ for all x, so the radius of

convergence is ∞.

42. $e^x = \sum\limits_{n=0}^{\infty} \dfrac{x^n}{n!}$ $\Rightarrow$ $10^x = e^{(\ln 10)x} = \sum\limits_{n=0}^{\infty} \dfrac{[(\ln 10)\, x]^n}{n!} = \sum\limits_{n=0}^{\infty} \dfrac{(\ln 10)^n x^n}{n!}$, $R = \infty$

43. $f(x) = \dfrac{1}{\sqrt[4]{16-x}} = \dfrac{1}{\sqrt[4]{16(1 - x/16)}} = \dfrac{1}{\sqrt[4]{16}\,(1 - \frac{1}{16}x)^{1/4}} = \frac{1}{2}\left(1 - \frac{1}{16}x\right)^{-1/4}$

$= \dfrac{1}{2}\left[1 + \left(-\dfrac{1}{4}\right)\left(-\dfrac{x}{16}\right) + \dfrac{\left(-\frac{1}{4}\right)\left(-\frac{5}{4}\right)}{2!}\left(-\dfrac{x}{16}\right)^2 + \dfrac{\left(-\frac{1}{4}\right)\left(-\frac{5}{4}\right)\left(-\frac{9}{4}\right)}{3!}\left(-\dfrac{x}{16}\right)^3 + \cdots\right]$

$= \dfrac{1}{2} + \sum\limits_{n=1}^{\infty} \dfrac{1 \cdot 5 \cdot 9 \cdot \cdots \cdot (4n-3)}{2 \cdot 4^n \cdot n! \cdot 16^n} x^n = \dfrac{1}{2} + \sum\limits_{n=1}^{\infty} \dfrac{1 \cdot 5 \cdot 9 \cdot \cdots \cdot (4n-3)}{2^{6n+1}\, n!} x^n$

for $\left|-\dfrac{x}{16}\right| < 1$ $\Leftrightarrow$ $|x| < 16$, so $R = 16$.

44. $(1 - 3x)^{-5} = \sum\limits_{n=0}^{\infty} \binom{-5}{n}(-3x)^n = 1 + (-5)(-3x) + \dfrac{(-5)(-6)}{2!}(-3x)^2 + \dfrac{(-5)(-6)(-7)}{3!}(-3x)^3 + \cdots$

$\qquad = 1 + \sum\limits_{n=1}^{\infty} \dfrac{5 \cdot 6 \cdot 7 \cdots \cdot (n+4) \cdot 3^n\, x^n}{n!}$ for $|-3x| < 1$ $\Leftrightarrow$ $|x| < \frac{1}{3}$, so $R = \frac{1}{3}$.

45. $e^x = \sum\limits_{n=0}^{\infty} \dfrac{x^n}{n!}$, so $\dfrac{e^x}{x} = \dfrac{1}{x}\sum\limits_{n=0}^{\infty}\dfrac{x^n}{n!} = \sum\limits_{n=0}^{\infty}\dfrac{x^{n-1}}{n!} = x^{-1} + \sum\limits_{n=1}^{\infty}\dfrac{x^{n-1}}{n!} = \dfrac{1}{x} + \sum\limits_{n=1}^{\infty}\dfrac{x^{n-1}}{n!}$ and

$\displaystyle\int \dfrac{e^x}{x}\,dx = C + \ln|x| + \sum\limits_{n=1}^{\infty}\dfrac{x^n}{n \cdot n!}.$

46. $(1 + x^4)^{1/2} = \sum\limits_{n=0}^{\infty}\binom{\frac{1}{2}}{n}(x^4)^n = 1 + \left(\frac{1}{2}\right)x^4 + \dfrac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2!}(x^4)^2 + \dfrac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3!}(x^4)^3 + \cdots$

$\qquad = 1 + \frac{1}{2}x^4 - \frac{1}{8}x^8 + \frac{1}{16}x^{12} - \cdots$

so $\int_0^1 (1 + x^4)^{1/2}\,dx = \left[x + \frac{1}{10}x^5 - \frac{1}{72}x^9 + \frac{1}{208}x^{13} - \cdots\right]_0^1 = 1 + \frac{1}{10} - \frac{1}{72} + \frac{1}{208} - \cdots.$

This is an alternating series, so by the Alternating Series Test, the error in the approximation

$\int_0^1 (1 + x^4)^{1/2}\,dx \approx 1 + \frac{1}{10} - \frac{1}{72} \approx 1.086$ is less than $\frac{1}{208}$, sufficient for the desired accuracy.
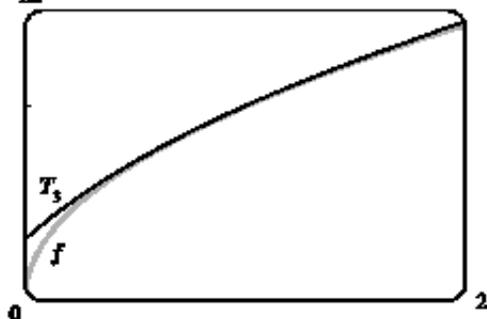
Thus, correct to two decimal places, $\int_0^1 (1 + x^4)^{1/2}\,dx \approx 1.09.$

47. (a)

n	$f^{(n)}(x)$	$f^{(n)}(1)$
0	$x^{1/2}$	1
1	$\frac{1}{2}x^{-1/2}$	$\frac{1}{2}$
2	$-\frac{1}{4}x^{-3/2}$	$-\frac{1}{4}$
3	$\frac{3}{8}x^{-5/2}$	$\frac{3}{8}$
4	$-\frac{15}{16}x^{-7/2}$	$-\frac{15}{16}$
$\vdots$	$\vdots$	$\vdots$

$$\sqrt{x} \approx T_3(x) = 1 + \frac{1/2}{1!}(x-1) - \frac{1/4}{2!}(x-1)^2 + \frac{3/8}{3!}(x-1)^3$$
$$= 1 + \tfrac{1}{2}(x-1) - \tfrac{1}{8}(x-1)^2 + \tfrac{1}{16}(x-1)^3$$

(b)

(c) $|R_3(x)| \leq \dfrac{M}{4!}|x-1|^4$, where $\left|f^{(4)}(x)\right| \leq M$ with

$f^{(4)}(x) = -\frac{15}{16}x^{-7/2}$. Now $0.9 \leq x \leq 1.1 \;\Rightarrow$

$-0.1 \leq x - 1 \leq 0.1 \quad\Rightarrow\quad (x-1)^4 \leq (0.1)^4$,

and letting $x = 0.9$ gives $M = \dfrac{15}{16(0.9)^{7/2}}$, so

$$|R_3(x)| \leq \frac{15}{16(0.9)^{7/2}\,4!}(0.1)^4 \approx 0.000\,005\,648$$

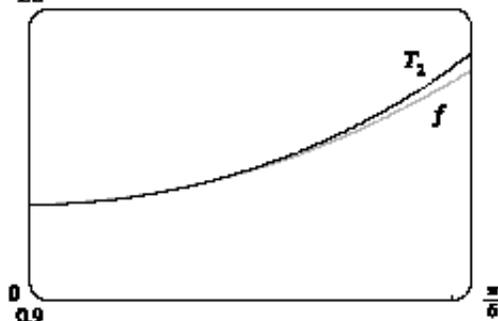$$\approx 0.000\,006 = 6 \times 10^{-6}$$

(d)
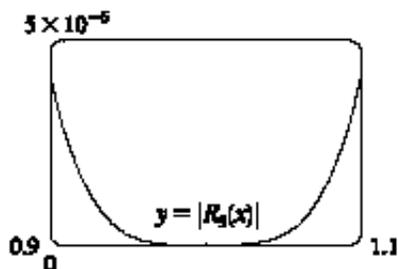
From the graph of $|R_3(x)| = |\sqrt{x} - T_3(x)|$, it appears

that the error is less than 5×10^{-6} on $[0.9, 1.1]$.

48. (a)

$\sec x \approx T_2(x) = 1 + \frac{1}{2}x^2$

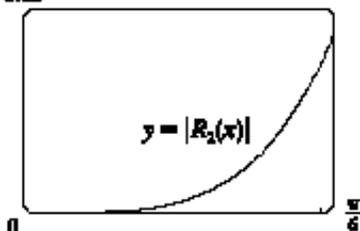| n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$\sec x$	1
1	$\sec x \tan x$	0
2	$\sec x \tan^2 x + \sec^3 x$	1
3	$\sec x \tan^3 x + 5 \sec^3 x \tan x$	0
$\vdots$	$\vdots$	$\vdots$

(b) 1.2

(c) $|R_2(x)| \leq \dfrac{M}{3!}|x|^3$, where $\left| f^{(3)}(x) \right| \leq M$ with

$f^{(3)}(x) = \sec x \tan^3 x + 5 \sec^3 x \tan x$. Now $0 \leq x \leq \frac{\pi}{6}$ $\Rightarrow$

$x^3 \leq \left(\frac{\pi}{6}\right)^3$, and letting $x = \frac{\pi}{6}$ gives $M = \frac{14}{3}$, so

$|R_2(x)| \leq \frac{14}{3 \cdot 6}\left(\frac{\pi}{6}\right)^3 \approx 0.111648$.

(d) 0.02

From the graph of $|R_2(x)| = |\sec x - T_2(x)|$, it appears that the

error is less than 0.02 on $\left[0, \frac{\pi}{6}\right]$.

49. $\sin x = \sum\limits_{n=0}^{\infty} (-1)^n \dfrac{x^{2n+1}}{(2n+1)!} = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \dfrac{x^7}{7!} + \cdots$, so $\sin x - x = -\dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \dfrac{x^7}{7!} + \cdots$ and

$\dfrac{\sin x - x}{x^3} = -\dfrac{1}{3!} + \dfrac{x^2}{5!} - \dfrac{x^4}{7!} + \cdots$. Thus, $\lim\limits_{x \to 0} \dfrac{\sin x - x}{x^3} = \lim\limits_{x \to 0}\left(-\dfrac{1}{6} + \dfrac{x^2}{120} - \dfrac{x^4}{5040} + \cdots\right) = -\dfrac{1}{6}$.

50. (a) $F = \dfrac{mgR^2}{(R+h)^2} = \dfrac{mg}{(1+h/R)^2} = mg \displaystyle\sum_{n=0}^{\infty} \binom{-2}{n} \left(\dfrac{h}{R}\right)^n$ (Binomial Series)

(b) We expand $F = mg \left[1 - 2\,(h/R) + 3\,(h/R)^2 - \cdots\right]$.

This is an alternating series, so by the Alternating Series

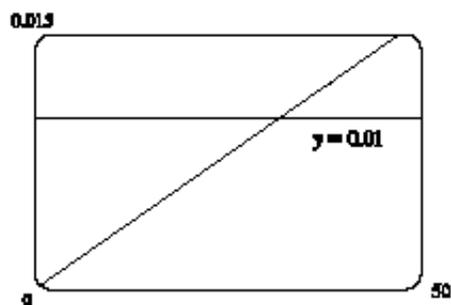Estimation Theorem, the error in the approximation $F = mg$ is less

than $2mgh/R$, so for accuracy within 1% we want

$\left|\dfrac{2mgh/R}{mgR^2/(R+h)^2}\right| < 0.01 \quad \Leftrightarrow \quad \dfrac{2h\,(R+h)^2}{R^3} < 0.01.$ This

inequality would be difficult to solve for h, so we substitute

$R = 6{,}400$ km and plot both sides of the inequality. It appears that

the approximation is accurate to within 1% for $h < 31$ km.

51. (a) From Formula 14a in Appendix C, with $x = y = \theta$, we get $\tan 2\theta = \dfrac{2\tan\theta}{1-\tan^2\theta}$, so $\cot 2\theta = \dfrac{1-\tan^2\theta}{2\tan\theta} \quad \Rightarrow$

$2\cot 2\theta = \dfrac{1-\tan^2\theta}{\tan\theta} = \cot\theta - \tan\theta$. Replacing θ by $\frac{1}{2}x$, we get $2\cot x = \cot\frac{1}{2}x - \tan\frac{1}{2}x$, or

$\tan\frac{1}{2}x = \cot\frac{1}{2}x - 2\cot x$.

(b) From part (a) with $\dfrac{x}{2^{n-1}}$ in place of x, $\tan\dfrac{x}{2^n} = \cot\dfrac{x}{2^n} - 2\cot\dfrac{x}{2^{n-1}}$, so the nth partial sum of $\displaystyle\sum_{n=1}^{\infty} \dfrac{1}{2^n}\tan\dfrac{x}{2^n}$ is

$$s_n = \dfrac{\tan(x/2)}{2} + \dfrac{\tan(x/4)}{4} + \dfrac{\tan(x/8)}{8} + \cdots + \dfrac{\tan(x/2^n)}{2^n}$$

$$= \left[\dfrac{\cot(x/2)}{2} - \cot x\right] + \left[\dfrac{\cot(x/4)}{4} - \dfrac{\cot(x/2)}{2}\right] + \left[\dfrac{\cot(x/8)}{8} - \dfrac{\cot(x/4)}{4}\right] + \cdots$$

$$+ \left[\dfrac{\cot(x/2^n)}{2^n} - \dfrac{\cot(x/2^{n-1})}{2^{n-1}}\right] = -\cot x + \dfrac{\cot(x/2^n)}{2^n} \quad \text{[telescoping sum]}$$

Now $\dfrac{\cot(x/2^n)}{2^n} = \dfrac{\cos(x/2^n)}{2^n\sin(x/2^n)} = \dfrac{\cos(x/2^n)}{x} \cdot \dfrac{x/2^n}{\sin(x/2^n)} \to \dfrac{1}{x} \cdot 1 = \dfrac{1}{x}$ as $n \to \infty$ since $x/2^n \to 0$

for $x \neq 0$. Therefore, if $x \neq 0$ and $x \neq k\pi$ where k is any integer, then

$$\sum_{n=1}^{\infty} \dfrac{1}{2^n}\tan\dfrac{x}{2^n} = \lim_{n\to\infty} s_n = \lim_{n\to\infty}\left(-\cot x + \dfrac{1}{2^n}\cot\dfrac{x}{2^n}\right) = -\cot x + \dfrac{1}{x}$$

If $x = 0$, then all terms in the series are 0, so the sum is 0.

52. We use the problem-solving strategy of taking cases:

Case (i):   If $|x| < 1$, then $0 \le x^2 < 1$, so $\lim\limits_{n \to \infty} x^{2n} = 0$ (see Example 9 in Section 8.1)
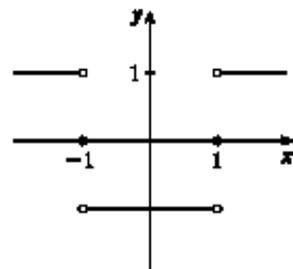
and $f(x) = \lim\limits_{n \to \infty} \dfrac{x^{2n} - 1}{x^{2n} + 1} = \dfrac{0 - 1}{0 + 1} = -1.$

Case (ii):   If $|x| = 1$, that is, $x = \pm 1$, then $x^2 = 1$, so $f(x) = \lim\limits_{n \to \infty} \dfrac{x^{2n} - 1}{x^{2n} + 1} = \lim\limits_{n \to \infty} \dfrac{1 - 1}{1 + 1} = 0.$

Case (iii):   If $|x| > 1$, then $x^2 > 1$, so $\lim\limits_{n \to \infty} x^{2n} = \infty$ and

$f(x) = \lim\limits_{n \to \infty} \dfrac{x^{2n} - 1}{x^{2n} + 1} = \lim\limits_{n \to \infty} \dfrac{1 - \left(1/x^{2n}\right)}{1 + \left(1/x^{2n}\right)} = \dfrac{1 - 0}{1 + 0} = 1.$

Thus, $f(x) = \begin{cases} 1 & \text{if } x < -1 \\ 0 & \text{if } x = -1 \\ -1 & \text{if } -1 < x < 1 \\ 0 & \text{if } x = 1 \\ 1 & \text{if } x > 1 \end{cases}$

The graph shows that f is continuous everywhere except at $x = \pm 1$.