Math 1b Final Review - Series

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Resources for Review

* Review Guides
  http://www.courses.fas.harvard.edu/~math1b/exams/
* Exams and solutions from previous years http://www.courses.fas.harvard.edu/~math1b/prevexams/
* Solutions to the Chapter Review Exercises http://www.courses.fas.harvard.edu/~math1a/exams/

Exam Particulars

* Tuesday, May 23 at 2:15-5:15 PM in Geology Lecture Hall
* No calculators allowed
* All out-of-sequence exams must be approved by the Final Exams Office

Geometric Series

Let

$$\sum_{k=0}^{\infty} ar^k = a + ar + ar^2 + ar^3 + \cdots .$$

- If $|r| < 1$, then $\displaystyle\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}$.
- If $|r| \geq 1$, the series diverges.

An Application

Suppose that \$100 is initially deposited in a bank. Experience has shown bankers that that on the average only 8% of the money deposited is withdrawn by the owner at any time. Consequently, bankers feel free to lend out 92% of their deposits. Thus, \$92 of the original \$100 is loaned out to other customers (to start a business for example). This \$92 will become someone else's income and, sooner or later, will be redeposited in the bank. Then 92% of the \$92 or \$84.64 is loaned out and eventually redeposited. Of the \$84.64, the bank loans out 92%, and so on. What is the total amount of money deposited in the bank as a result of these transactions?

Series

- $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^4+n+1}$

- $\displaystyle\sum_{n=0}^{\infty} \left(\frac{1}{2^n}+\frac{1}{3^n}\right)^2$

- $\displaystyle\sum_{n=1}^{\infty} \frac{\sin(1/n)}{n^3}$

- $\displaystyle\sum_{n=2}^{\infty} \frac{2^n}{n!(\ln n)^3}$

1. If the series is of the form $\sum 1/n^p$, then it is a p-series. The series converges for $p > 1$ and diverges for $p \leq 1$.

2. If the series has the form $\sum ar^n$, then it is a geometric series and converges for $|r| < 1$ and diverges for $|r| \geq 1$.

3. If the series is similar to a p-series or a geometric series, consider the Comparison Test.

4. If $\lim_{n\to\infty} a_n \neq 0$, then the series diverges.

5. If the series is of the form $\sum(-1)^{n+1}a_n$, consider applying the Alternating Series Test. You can also test for absolute convergence.

6. If the series involves products, factorials, or constants raised to the nth power, consider the Ratio Test.

7. If $a_n = f(n)$ and the integral $\int_1^\infty f(x)\,dx$ is easily evaluated, the Integral Test may be useful assuming the hypothesis of the test are satisfied.

8. Is the series a telescopic series? If so, convergence or divergence can be determined by computing the limit of the partial sums of the series.

Alternating Series

Let

$$\sum_{n=1}^{\infty}(-1)^{n+1}a_n = a_1 - a_2 + a_3 - a_4 + \cdots$$

satisfy the following conditions.

1. $a_1 \geq a_2 \geq a_3 \geq \cdots$

2. $\lim_{n\to\infty} a_n = 0$

Then the series converges and

$$|R_n| = |S - S_n| \leq a_{n+1}.$$

Absolute and Conditional Convergence

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln n}$$

Power Series Representations

Find the interval of convergence of the power series

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{k+1}(x-4)^k$$

Important Power Series

Function	Series	Interval of Convergence
e^x	$1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots$	$(-\infty,\infty)$
$\sin x$	$x-\frac{x^3}{3!}+\frac{x^5}{5!}-\cdots$	$(-\infty,\infty)$
$\cos x$	$1-\frac{x^2}{2!}+\frac{x^4}{4!}-\cdots$	$(-\infty,\infty)$
$\frac{1}{1-x}$	$1+x+x^2+x^3+\cdots$	$(-1,1)$

Taylor Series

(a) Write down a power series expansion for e^{-x^3}.

(b) Write down a power series expansion for $\int e^{-x^3}\,dx$ and determine its radius of convergence.

(c) Use your answer in part (b) to find a series for $\int_0^{1/2} e^{-x^3}\,dx$.

(d) If you approximate the definite integral $\int_0^{1/2} e^{-x^3}\,dx$ by taking the partial sum consisting of the first four nonzero terms of the series that you obtained in part (c), what is the maximum error for your approximation.

Taylor Polynomials

Let f be a function having derivatives of all orders for all real numbers. The third-degree Taylor polynomial for f about $x = -2$ is given by

$$T_3(x) = 2 - \frac{3}{8}(x+2)^2 - \frac{1}{12}(x+2)^3.$$

(a) Find $f(-2)$, $f'(-2)$, and $f''(-2)$.

(b) Determine whether f has a local minimum, a local maximum, or neither at $x = -2$. Justify your answer.

(c) Use $T_3(x)$ to find an approximation for $f(0)$.

(d) The fourth derivative of f satisfies the inequality

$$\left| f^{(4)}(x) \right| \leq \frac{1}{4}$$

for all x in the closed interval $[-2, 0]$. Find an error bound on the approximation for $f(0)$ that you found in part (c).