

## REVIEW NOTES ON DIFFERENTIAL EQUATIONS

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### 1. BASIC CONCEPTS FOR DIFFERENTIAL EQUATIONS

A *differential equation* is an equating relating an unknown function to its derivative(s). It is easiest to give examples – in all of the following,  $y$  is a function of  $t$ :

$$(1) \quad dy/dt = \frac{1}{1+t^2}$$

$$(2) \quad dy/dt = 3y$$

$$(3) \quad dy/dt = -t/y$$

$$(4) \quad y'' + 3y' + 2y = 0$$

The *order* of a differential equation is the highest numbered derivative which appears in the equation. Equations (1) through (4) are first-order differential equations; equation (5) is a second-order differential equation. (By the way, one could certainly consider differential equations of higher order: e.g. the equation  $y^{(10)} = 0$  is a tenth-order differential equation which asks for all functions whose tenth derivative is identically zero. Any degree nine polynomial  $a_9t^9 + a_8t^8 + \dots + a_1t + a_0$  is a solution to this equation. But in our course we have had our hands full with first-order equations and very particular second-order equations.)<sup>1</sup> To say that a function  $y = y(t)$  is a *solution* to a differential equation just means that if we substitute it into both sides of the equation we do indeed get an equality. For example, the function  $y_1(t) = \arctan(t)$  is a solution to equation (1), because the derivative of  $\arctan(t)$  is  $\frac{1}{1+t^2}$ . It is not the only solution;  $y(t) = \arctan(t) + C$  also works.

**Existence and uniqueness of solutions (first order case):** The general first-order differential equation is  $dy/dt = f(t, y)$  where  $f$  is just any function of  $t$  and  $y$ . Notice that equations (1) through (4) are of this form – for (4), bring the  $y^2$  to the other side. A first-order differential equation never has just one solution but rather an *one-parameter family* of solutions. We specify a particular solution by

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<sup>1</sup>Notice that all our differential equations above are in the form: “ $y' =$  some function of  $y$  and  $t$ ” or “ $y'' =$  some function of  $y'$ ,  $y$  and  $t$ ”; in fact as part of our definition of a differential equation (which wasn't very precise) we should understand that we mean a differential equation of this form, or we could have problems:

For example: Show that there is *no* function  $y = y(t)$  such that  $(dy/dt)^2 + 1 = -(t^2 + y^2)$ . (Hint: look at the signs of the left hand side and the right hand side of the equation!)  
Solution : The left-hand side is strictly positive, and the right-hand side is less than or equal to zero. No way.

requiring the solution to take a certain value at a certain point; notice that this is a generalization of the situation of taking antiderivatives; in the differential equation  $dy/dt = f(t)$ , the general solution is  $y = F(t) + C$ , and we can pick out a particular choice of antiderivative by imposing a condition  $y(t_0) = y_0$ , or  $y_0 = F(t_0) + C$ , so that  $C = y_0 - F(t_0)$ . This leads to the following terminology:

(Initial value problem): A first-order initial value problem is a first-order differential equation together with an initial condition:

$$dy/dt = f(t, y), \quad y(t_0) = y_0$$

Now we have the theoretical result that we have exploited so many times in our study of differential equations:

**Theorem 1.** (*Existence and uniqueness theorem for first-order equations*) Any first order differential equation of the form  $dy/dt = f(t, y)$ , with initial condition  $y(t_0) = y_0$  where  $f(t, y)$  is well-behaved<sup>2</sup> has a unique solution  $y$ , i.e. there is a unique function  $y$  satisfying the differential equation and such that  $y(t_0) = y_0$ .

We can also view the collection of all solutions to a given first-order equation  $dy/dt = f(t, y)$  geometrically: we view each one as a *solution curve* in the  $(t, y)$ -plane. We can reinterpret the existence and uniqueness theorem as telling us that (i) every point on the  $(t, y)$ -plane lies on a solution curve, and (ii) no point lies on more than one solution curve. Equivalently, we say that the solution curves *fill up the plane and never cross*.

**Initial value problems for second-order differential equations:** To specify a particular solution to a second-order differential equation will require two initial conditions. Again, this can be predicted from the differential equation  $y'' = f(t)$  which asks for all functions  $\mathcal{F}$  whose second derivative is  $f$ . We can solve this equation in two steps:  $y' = F(t) + C_1$ , where  $F' = f$ , and  $y = \mathcal{F}(t) + C_1t + C_2$ , where  $\mathcal{F}' = F$ . Thus we have a modified

(Initial value problem): A second-order initial value problem is a second-order differential equation together with two initial conditions:

$$y'' = f(t, y, y'), \quad y(t_0) = y_0, \quad y'(t_0) = v_0$$

And just as for first-order equations we have the

**Theorem 2.** (*Existence and uniqueness theorem for second-order equations*) Any initial value problem  $y'' = f(t, y, y'), y(t_0) = y_0, y'(t_0) = v_0$  has a unique solution provided  $f$  is well-behaved.

## 2. TAXONOMY OF FIRST-ORDER DIFFERENTIAL EQUATIONS

We have studied several different kinds of first-order differential equations.

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<sup>2</sup>by well-behaved we mean that they will have infinitely many derivatives when regarded as either functions of  $t$  or  $y$ . All the functions we deal with in this course will be well-behaved.

**2.1. Integration equations.** An equation of the form

$$dy/dt = f(t)$$

can be thought of as an *integration equation*, because it simply asks for an indefinite integral of  $f(t)$ . I.e., if  $F' = f$  is any antiderivative, the general solution is  $y = F(t) + C$ ; note well that the solution curves are vertical translates of each other. Since solving integration equations requires no new techniques, we won't see many of them, but they are good to keep in mind as well-understood special cases to understand how the theory should go (e.g. from them we could guess the need for initial conditions).

Geometrically an "integration equation" says that the slope depends only upon  $t$ ; therefore the solutions are vertical translates.

**2.2. Autonomous equations.** An equation of the form

$$dy/dt = f(y)$$

is called an *autonomous equation*. Compare with the previous example: the function on the right-hand-side is only a function of  $y$ , and hence independent of time (= autonomous). What does the time-independence of solutions mean? It means that all solution curves passing through the same  $y$ -value  $y = y_0$  are governed by the same law:  $dy/dt = f(y_0)$ . Thus, if we take any solution curve  $y(t)$  and push it to the left or the right, what the differential equation is telling it to do at any particular  $y$ -value is not changing, and we find that solution curves to an autonomous equation are *invariant under horizontal translation*.<sup>1)</sup> Autonomous equations lend themselves well to a *qualitative analysis*:

Procedure for qualitative analysis of  $dy/dt = f(y)$  where  $f(y)$  is continuous:

Step 1: Find the equilibrium solutions, that is, the  $y$ -values  $y_i$  such that  $f(y_i) = 0$ . These constant solutions are the equilibrium solutions.

Step 2: Do a sign analysis of  $dy/dt = f(y)$ . In between equilibrium solutions,  $f(y)$ , hence  $dy/dt$ , has constant sign (we assume that  $f$  is a *continuous* function of  $y$ ), i.e. in "strips" between equilibrium values the solutions  $y(t)$  are always increasing or always decreasing. Test to see whether  $f(y)$  is positive or negative in that region to see whether solutions are increasing or decreasing.

Step 3: (Do this if you want information on concavity.) Do a sign-analysis of  $y'' = f'(y)dy/dt = f'(y)f(y)$  to determine the concavity. (Don't forget the chain rule. You'll notice that all the equilibrium solutions are  $y$ -values at which  $y'' = 0$ .) By testing the sign in between the zeroes of  $y''$  we learn when the solutions are concave up and when concave down and also the  $y$ -coordinates of the inflection points.

Commentary: The situation in the "bounded strips" – i.e. in the region between two equilibrium solutions  $y_1 < y_2$  – is very well-controlled. Indeed, if the solution is always increasing then all curves  $y(t)$  have  $\lim_{t \rightarrow -\infty} x(t) = y_1$  and  $\lim_{t \rightarrow +\infty} x(t) = y_2$  and if the solution is always decreasing then just the opposite is true: that is, the constant solutions above and below will be the limiting values at  $\pm\infty$  of all solution

curves in the strip between them. This forces the solution curves to change concavity at least once on the strip. It also forces the fact that every solution curve in a bounded strip is a horizontal translate of every other solution curve in the same strip (this follows from looking carefully at the picture and using the fact that solution curves fill up the strip and don't cross; in case you want the full argument, send email to pete@math.harvard.edu.)

In the unbounded strips, again the solutions are either always increasing or always decreasing. (In fact in the examples we see, it will most often be the case that the solution blows up in finite time; for instance if  $f(y)$  is a polynomial of degree at least 2 then this is what happens.) Autonomous Differential equations are separable and can be solved using separation of variables.

**2.3. Separable equations.** An equation of the form

$$dy/dt = g(t)h(y)$$

i.e. a function of  $t$  multiplied by a function of  $y$ , is called *separable*. In this case we can “cross-multiply” and integrate to get a solution of the equation:  $dy/h(y) = g(t)dt$ ,  $\int \frac{dy}{h(y)} = \int g(t)dt$ , and integrating both sides gives an equation of the form “some function of  $y$  is equal to some function of  $t$ ”. This method is called separation of variables; it is by far the most important solution technique in this course. Notice that on page 1, Equations (1) through (3) are separable, and equation (4) is not.

In particular every autonomous equation is separable ( $f(y)$  is of the form a function of  $y$  times 1, a function of  $t!$ ). Why then do we study autonomous equations? The answer is that if we are interested in *qualitative* information (like limiting behavior), doing the qualitative analysis of autonomous equations is often much easier than separating variables and solving. This is true especially when we separate variables and only get an implicitly defined function of  $y$ .

Example: Consider the differential equation  $dy/dt = (y - 1)(y - 3)$ . If  $y(t)$  is the unique solution with  $y(0) = 2$ , what is the limit of  $y$  as  $t$  approaches  $\infty$ ?

We'll solve this in two ways; first, using the qualitative methods of above, and second, by separating variables and explicitly solving. We trust that the reader will have no trouble seeing which turns out to be easier.

First solution: The equation  $dy/dt = (y - 1)(y - 3)$  has equilibrium solutions at  $y = 1$  and  $y = 3$ ; the initial value  $y(0) = 2$  puts us in the (unique) bounded strip in between these two equilibrium solutions. Since  $(2 - 1)(2 - 3) < 0$ , our solution curve is decreasing in this strip, hence asymptotic to its bottom equilibrium curve; that is  $\lim_{t \rightarrow \infty} y(t) = 1$ .

Second solution: Separating variables, we get  $\int \frac{dy}{(y-1)(y-3)} = \int dt = t + C$ . We must remember partial fractions:  $\frac{1}{(y-1)(y-3)} = \frac{A}{y-1} + \frac{B}{y-3}$  and we have to solve for constants  $A$  and  $B$ . Clearing denominators we get  $1 = (y - 3)A + (y - 1)B$  and plugging in  $y = 1$  and then  $y = 3$  we get  $A = -1/2$ ,  $B = 1/2$ . Thus the integral is  $1/2(\int \frac{dy}{y-3} - \int \frac{dy}{y-1}) = 1/2(\ln(y - 3) - \ln(y - 1)) = 1/2(\ln(\frac{y-3}{y-1})) = t + C_1$ , so

$\ln\left(\frac{y-3}{y-1}\right) = 2t + C_2$ , or  $\frac{y-3}{y-1} = C_3 e^{2t}$ , and we still are not done because we only have  $y$  implicitly. Putting  $g(y) = \frac{y-3}{y-1}$ , we need the function  $h(y)$  which is functionally inverse to  $g$ , i.e. such that  $h(g(y)) = y$ , and then  $y = h(Ce^{2t})$ . To invert a function interchange the variables, so  $t = (y-3)/(y-1)$ ,  $yt - t = y - 3$ ,  $y(t-1) = t-3$ , so  $y = (t-3)/(t-1)$ , and  $y = \frac{Ce^{2t}-3}{Ce^{2t}-1}$ . Using  $y(0) = 2$ , we get  $2 = \frac{C-3}{C-1}$ , and solving for  $C$  we get  $C = -1$ , so finally  $y = \frac{-e^{2t}-3}{-e^{2t}-1}$ . If we try to take the limit as  $t \rightarrow \infty$  we get  $\infty/\infty$ , so we can apply L'Hopital's rule, and as the numerator and the denominator differ only by a constant, their derivatives are the same, so the limit is  $\lim_{t \rightarrow \infty} x(t) = 1$ .

**2.4. linear equations.** An equation of the form

$$dy/dt + P(t)y = Q(t)$$

is called *first-order linear*. (For any fixed  $t$  it is a linear equation in  $y'$  and  $y$ .) Some first order linear differential equations are actually separable and can be most easily solved by separating variables. We can solve such equations via the method of *integrating factors* (or, if you like, via a shameless trick). Let  $v(t) := e^{\int P(t) dt}$ ; this is our integrating factor. Now, multiply by  $v$  we get

$$vdy/dt + vPy = vQ(t)$$

The left-hand side is precisely  $(vy)'$ . (Check this for yourself.) Therefore we can rewrite the equation as

$$(vy)' = vQ(t)$$

and just integrate to get the solution:

$$vy = \int vQ(t)dt + C$$

$$y = \frac{\int vQ(t)dt + C}{v} = \frac{\int vQ(t)dt}{v} + C/v$$

**Problem 1:** For each of the following first-order differential equations, decide whether the equation is an 'integration equation' (i.e. of the form  $dy/dt = f(t)$ ), or an autonomous equation or neither. If it is an integration equation, solve it; if it is autonomous, do a qualitative analysis. There will be two that are neither. If a remaining differential equation is separable, separate variables and solve. Otherwise, see if it is linear and if so, solve.

(3a)  $dy/dt = 5y + te^t - 5y$ .

(3b)  $dy/dt = \cos y$ .

(3c)  $dy/dt = y + e^t$ .

(3d)  $dy/dt = \frac{\sec(y)}{4t^2}$ .

(3e)  $dy/dt = \sin y + 1$ .

### 3. DIRECTION FIELDS

Many first-order equations do not fall under the special cases described in the preceding section: a first order differential equation can be neither separable nor linear. What if we work for NASA and such an equation is giving us the motion of a rocket if we attach a certain type of engine – i.e., what if we really, really want to

know something about the solution to a differential equation that we do not know how to solve exactly? There is a numerical method due to Leonhard Euler (arguably the greatest mathematician of the 18th century and indisputably the most prolific mathematician of all time) that, given an initial condition and a lot of computation, will tell us approximately where we are going to end up at some later time.

The idea is very simple: we view the differential equation  $dy/dt = f(t, y)$  as assigning to each point in the plane a *slope* or *direction*. This “direction field” has the property that any solution curve passing through a point has the slope of the tangent line (the derivative!) equal to the slope at that point. Thus, if you imagine a cast plane with lots of little slope elements on the ground, then as you walk from left to right along the plane, the slope elements tell you whether to go up or down; that is, they push you along the solution curve passing through your initial point. The caveat is that you have to make a decision: since the tangent line is the *infinitesimal* rate of change at the point in question, it is strictly speaking incorrect to follow it along for any finite amount of time (we should switch continuously from one direction to another), but that’s too bad: we cannot follow the path for an instantaneous amount of time. Instead therefore we choose a *step size*  $\Delta t$  and agree to follow each slope element for time  $t$  before looking down to change our direction. We do this until we get to some prearranged time  $t_{\max}$ , and the  $y$ -coordinate of where we end up is the *approximate* value of the solution curve  $y(t_{\max})$ .

We didn’t study Euler’s method (it is best done with a computer) but you should be familiar with slope fields and matching slope fields with differential equations.

#### 4. PHASE PLANE ANALYSIS OF FIRST-ORDER SYSTEMS

In this section we look at systems of differential equations of the form

$$\begin{aligned}\frac{dx}{dt} &= f(x, y) \\ \frac{dy}{dt} &= g(x, y)\end{aligned}$$

where  $f$  and  $g$  are well-behaved continuous functions. Instead of trying to solve for  $y$  in terms of  $t$  and  $x$  in terms of  $t$  (we *can* sometimes do this – as you’ll see in a problem below) we’ll analyze the relationship in the  $xy$ -plane where we can use a directed trajectory (a curve with an arrow on it indicating the direction we travel along it) to illustrate how  $x$  and  $y$  change with  $t$ , which we’ll think of as time. This sort of analysis is called **phase plane analysis**. For each point  $P = (x(0), y(0))$  in the  $xy$ -plane there is a trajectory through  $P$ . If  $f(x, y)$  and  $g(x, y)$  are sufficiently well-behaved (they will always be so in this course) then there is a unique trajectory through  $P$ . Think of  $(x(t), y(t))$  as tracing out a path (trajectory) in the plane as  $t$  increases. We’d like to know the possible behaviours of this path. Notice that if the point  $P$  is an equilibrium point then as  $t$  increases  $(x(t), y(t))$  remains fixed at  $P$  and the trajectory is just a point in the plane.

We can use such systems to model situations in which the rate of change of  $x$  and  $y$  depend upon the amounts of both  $x$  and  $y$ . For example, we can predator/prey relations, symbiotic relations, competitive relationships... or we could model the evolution of a disease. Steve Strogatz (formerly of Harvard and then MIT) used systems of differential equations to model a love/hate relationship between ‘Romeo’

and ‘Juliet’ and his model made the local papers! (You will get a chance to work through this model in a problem below. And I (Robin) had better claim authorship to this section just in case the example is too silly for Pete!) To interpret the model just by looking at the equations, first look at what happens to  $x$  in the absence of  $y$  (that is, when  $y = 0$ ) and what happens to  $y$  in the absence of  $x$ . Then see how the presence of  $y$  affects  $x$  and vice versa. We expect you to be able to look at a system of equations and be able to distinguish a predator/prey relation from a competitive one from a symbiotic relationship for instance.

Here is how the actual phase plan analysis works:

- Step 1. If your variables are not  $x$  and  $y$ , decide which will play the role of  $x$  and which the role of  $y$  and label your axes accordingly. (Any way you do this is fine, but it must be done up front and then not changed.)
- Step 2. Draw the null clines.

Draw the  $y$ -null clines, the curves (often lines) on which  $\frac{dy}{dt} = 0$ . Draw horizontal slope marks all along the  $y$ -null clines to show that this is where  $\frac{dy}{dt} = 0$ . (If  $\frac{dy}{dt} = 0$  then when a trajectory crosses this null cline it does so with a horizontal tangent since  $y$  is not changing. It may help you to keep in mind that  $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$ ; we’ve just identified places where the numerator is zero.)

Draw the  $x$ -null clines, the curves on which  $\frac{dx}{dt} = 0$ . Draw vertical slope marks all along the  $x$ -null clines to show that this is where  $\frac{dx}{dt} = 0$ . (If  $\frac{dx}{dt} = 0$  then when a trajectory crosses this null cline it does so with a vertical tangent since  $x$  is not changing. It may help you to keep in mind that  $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$ ; we’ve just identified places where the denominator is zero.)

- Step 3. Identify all the equilibrium points. These are the points where a  $y$ -null cline and an  $x$ -null cline intersect: at these points and only at these points do we have  $\frac{dy}{dt}$  and  $\frac{dx}{dt}$  both equal to zero.
- Step 4. Orient the slope lines on the null clines.

Orient the horizontal slope marks. The question is “left, or right?”, or, in other words, does  $x$  increase or does  $x$  decrease with time. The only way to tell is to look at  $\frac{dx}{dt}$ . Where  $\frac{dx}{dt} > 0$  the arrow goes to the right because  $x$  is increasing. Where  $\frac{dx}{dt} < 0$  the arrow goes to the left because  $x$  is decreasing. The orientation can change from left to right only across an equilibrium point. (Think about why.)<sup>3</sup>

Orient the vertical slope marks. The question is “up, or down?”, or, in other words, does  $y$  increase or does  $y$  decrease with time. The only way to tell is to look at  $\frac{dy}{dt}$ . Where  $\frac{dy}{dt} > 0$  the arrow goes up because  $y$  is increasing. Where  $\frac{dy}{dt} < 0$  the arrow goes down because  $y$

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<sup>3</sup>The  $y$ -null cline is determined by the equation  $\frac{dy}{dt} = g(x, y)$ . The direction of the arrows (left/right) is determined by the sign of  $\frac{dx}{dt} = f(x, y)$ . I mention this because some students found that confusing. The best way to avoid confusion is to simply concentrate on the question you’re trying to answer. Left/right is a question of  $x$  increasing or decreasing and can only be determined by  $\frac{dx}{dt}$ .

is decreasing. The orientation can change from up to down only across an equilibrium point. (Think about why.)

- Step 5. The null clines have divided the phase plane into regions on which the general direction of the trajectory doesn't change. Determine the direction in each of the regions.

If  $\frac{dx}{dt} > 0$  and  $\frac{dy}{dt} > 0$ , then both  $x(t)$  and  $y(t)$  are increasing so the trajectory must be traveled up and to the right.

If  $\frac{dx}{dt} > 0$  and  $\frac{dy}{dt} < 0$ , then  $x(t)$  is increasing and  $y(t)$  is decreasing so the trajectory must be traveled down and to the right.

And so on.

- Step 6. If the system is simple enough, and if  $\frac{dy}{dx} = \frac{g(x,y)}{f(x,y)}$  is separable, then sometimes solving this differential equation can be useful. (You will see this in the Romeo and Juliet problem below.)

Use this phase plane analysis to see what you can say about the short and long term behavior of the system, keeping in mind that through any point in the plane there is a unique trajectory. Sometimes the analysis is fairly straightforward. Other times it is not. We do not expect you to be able to distinguish between trajectories that are closed curves versus those that are spirals unless solving for  $y$  in terms of  $x$  is quite simple (as in the Romeo and Juliet problem.)

To summarize, the analysis proceeds as follows.

- Step 1. If your variables are not  $x$  and  $y$ , decide which will play the role of  $x$  and which the role of  $y$  and label your axes accordingly.
- Step 2. Draw the null clines.
- Step 3. Identify all the equilibrium points.
- Step 4. Orient the slope lines on the null clines.
- Step 5. The null clines have divided the phase plane into regions on which the general direction of the trajectory doesn't change. Determine the direction in each of the regions.

- Step 6. If the system is simple enough, and if  $\frac{dy}{dx} = \frac{g(x,y)}{f(x,y)}$  is separable, then sometimes solving can be useful.

**Problem 2:** Let  $j(t)$  be a measure of Juliet's feelings for Romeo and  $r(t)$  be a measure of Romeo's feelings for Juliet. Positive values of  $j$  or  $r$  indicate love and negative values indicate hate. In Steve Strogatz's version of the lovers' tale the relationship between Romeo and Juliet can be modelled by

$$\begin{aligned}\frac{dj}{dt} &= -r \\ \frac{dr}{dt} &= j\end{aligned}$$

- (a) In this model which of the following statements is true? (Use the math to answer, but be on the lookout that this problem was written by a man and that a woman probably would have used a different model!! (RG))
  - (i) The more Romeo loves Juliet the more strongly Juliet falls for Romeo.
  - (ii) The more Juliet loves Romeo the more strongly Romeo falls for her.
  - (iii) Juliet pines for Romeo most when he is thinking of leaving her, but when his affections turn towards her she turns away.
- (b) Let  $j$  play the role of  $y$  and  $r$  play the role of  $x$ . Do a phase plane analysis. Sketch the  $r$ - and  $j$ - nullclines in the  $rj$ - plane. Find all equilibrium

points of the system. Then orient the nullclines and orient the regions of the plane. (Note: don't restrict yourself to the first quadrant. You'll use all four quadrants.)

- (c) Find  $\frac{dj}{dr}$  in terms of  $j$  and  $r$ . This is a separable differential equation. Solve it. Then draw the trajectories in the  $rj$ -plane. Orient them using the arrows you drew in part (b).
- (d) Suppose that  $r(0) = 1$  and  $j(0) = 0$ . What happens in the short run? In the long run? Indicate the trajectory in the phase-plane drawing you made.
- (e) We know  $\frac{dj}{dt} = -r$  so  $\frac{d^2j}{dt^2} = -\frac{dr}{dt} = -j$ . Therefore

$$\frac{d^2j}{dt^2} = -j.$$

Similarly, we can find an expression for  $\frac{d^2r}{dt^2}$  that is a second order homogeneous, linear differential equation with constant coefficients. Solve these equations and use the initial conditions given in (d) to find  $j(t)$  and  $r(t)$  respectively. Graph  $j(t)$  and  $r(t)$ . Is this consistent with your phase-plane picture?

## 5. LINEAR HOMOGENEOUS SECOND-ORDER DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

In this section we study the second-order differential equation

$$y'' + by' + cy = 0$$

which we call *linear, homogeneous with constant coefficients*. It is an interesting equation because depending upon the constants the solutions exhibit completely different behavior (sometimes oscillatory, sometimes not).

**5.1. from specific to general solutions.** Recall that we are looking for a two-parameter family of solutions to this second-order differential equation. It turns out that since the equation is homogeneous with constant-coefficients, the passage from two particular solutions to the general solution is rather easy. If  $y_1(t), y_2(t)$  are any two solutions to the equation  $y'' + by' + cy = 0$ , then also  $y_1(t) + y_2(t)$  is a solution, and if  $C$  is any constant, then also  $Cy_1(t)$  is a solution. We can check this directly: by assumption  $y_1'' + by_1' + cy_1 = y_2'' + by_2' + cy_2 = 0$ , so

$$\begin{aligned} (y_1 + y_2)'' + b(y_1 + y_2)' + a_0(y_1 + y_2) &= \\ y_1'' + y_2'' + by_1' + by_2' + cy_1 + cy_2 &= \\ (y_1'' + by_1' + cy_1) + (y_2'' + by_2' + cy_2) &= 0 + 0 = 0 \end{aligned}$$

The derivation that  $y_1$  a solution implies  $Cy_1$  is a solution is very similar. Putting these two facts together and appealing to the uniqueness theorem for second-order differential equations we get:

**Theorem 3.** *Let  $y_1(t), y_2(t)$  be two independent solutions to the equation  $y'' + by' + cy = 0$ , where independent means that  $y_2 \neq Cy_1$  for any constant  $C$ . Then the general solution is*

$$C_1y_1 + C_2y_2$$

for arbitrary constants  $C_1, C_2$ .

Because of this result, to solve our equation we just need to produce (by any means necessary!) any two particular solutions which are not multiples of each other; then multiplying by arbitrary constants and adding we will get the general solution. Suppose we guess that solutions might look like  $y = e^{rt}$ . If we try this in the differential equation  $y'' + by' + cy = 0$  we find that  $y = e^{rt}$  is a solution if and only if  $r^2 + br + c = 0$ . (Try this for yourself and see.) We call

$$r^2 + br + c = 0$$

the *characteristic equation*. The roots of the characteristic equation give us the data we need to write down the general solution. Recall that when we look at the roots of a quadratic polynomial there are three different things that can happen: formally, the quadratic formula tells us:

$$r = \frac{-b \pm \sqrt{b^2 - 4c}}{2}$$

Put  $D := b^2 - 4c$ , the *discriminant* of the quadratic polynomial (so-called because it discriminates between various cases). Notice especially that in the quadratic formula we need to take a square root, so we need to worry about the sign of  $D$ . Thus the three cases correspond to:  $D > 0$ ,  $D = 0$  and  $D < 0$ .

$D > 0$  case: we can thus take the square root in the quadratic formula and we get two distinct real roots  $r_1 \neq r_2$ .

$D = 0$  case: we can take the square root of zero, but  $\pm 0 = 0$  and there is only one root:  $-b/2$ ; we call this the repeated root case.

$D < 0$  case: If  $x$  is any real number, then  $x^2 \geq 0$ , so there is no real number whose square is  $D$ . To insist that we have a solution to every quadratic polynomial we need to introduce the complex numbers, of the form  $\alpha + \beta i$ , where  $i^2 = -1$ . Then e.g. the equation  $x^2 + D = 0$  has the solution  $x = \pm\sqrt{D} = \pm\sqrt{-1}\sqrt{|D|} = \pm\sqrt{|D|}i$ .

So in the  $D < 0$  case we will be getting two complex roots,  $r = \frac{-b \pm \sqrt{|D|}i}{2}$  which are complex conjugates of each other: complex numbers of the form  $\alpha + \beta i, \alpha - \beta i$  are said to be complex conjugates; they have the same “real part”  $\alpha$  but opposite “imaginary parts”  $\beta, -\beta$ .

Case 1 ( $D > 0$ ): we have  $r_1$  and  $r_2$  are distinct real roots, and our strategy works perfectly: indeed  $e^{r_1 t}$  and  $e^{r_2 t}$  are two different solutions, so the general solution is

$$y(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}$$

Case 2 ( $D = 0$ ): Here  $r_1 = r_2$ , so  $e^{r_1 t} = e^{r_2 t}$  and we’ve found one solution; we still need to find another one. The solution  $y_2 = t e^{r_1 t}$  also works. You can show this using the fact that if  $D=0$  then  $r = -b/2$ . The general solutions is

$$y(t) = C_1 e^{r_1 t} + C_2 t e^{r_1 t}.$$

Case 3: This is where things get a little wild. Notice that in each of the two previous cases there was always at least one purely exponential solution  $y(t) = e^{r_1 t}$ . But consider the equation  $y'' + y = 0$ ; this asks for functions which are the opposite of their second derivative. Sine and cosine famously have this property; by

Theorem 2 the general solution to this equation *must* be  $C_1 \cos t + C_2 \sin t$  – no exponentials here! And there is a second problem: our characteristic polynomial has complex roots, so that our strategy is telling us to consider the apparently ridiculous  $e^{(a+bi)t}$ ,  $e^{(a-bi)t}$  as two solutions to our equation.

But now a mathematical miracle occurs: these two problems cancel themselves out. Namely, over the complex numbers exponentials are intimately related to sines and cosines; we have

**Theorem 4.** (*Euler's formula*) For all real  $t$ ,  $e^{it} = \cos t + i \sin t$ .

Remark: This is a nice snapshot of Euler's mathematical genius, which was to guess and then prove astounding identities in the realm of functions and series. In all of history his only rival at this was the early 20th century Indian mathematician Srinivasa Ramanujan.

Proof: As a preparation, we should understand the powers of  $i$ :  $i^2 = -1$ ,  $i^3 = i \cdot i^2 = i \cdot (-1) = -i$  and  $i^4 = (i^2)^2 = (-1)^2 = 1$ ; since we came back where we started this means that any power of  $i$  is going to be  $1, i, -1$  or  $-i$ ; moreover the even powers of  $i$  will alternate between  $\pm 1$  and the odd powers of  $i$  will alternate between  $\pm i$ . With this in mind we separate

$$e^{it} = 1 + (it) + (it)^2/2! + (it)^3/3! + (it)^4/4! + (it)^5/5! + \dots$$

into even and odd powers, i.e. into

$$\begin{aligned} & [1 + i^2 t^2/2! + i^4 t^4/4! + i^6 t^6/6! + \dots] + [(it + i^3 t^3/3! + i^5 t^5/5! + \dots)] \\ & [1 - t^2/2! + t^4/4! - \dots + (-1)^n t^{2n}/(2n)! + \dots] + i[t - t^3/3! + t^5/5! - \dots + (-1)^n t^{2n+1}/(2n+1)! + \dots] \\ & = \cos t + i \sin t \end{aligned}$$

as promised.

This gives us hope to carry through our strategy even in Case 3: we might get sines and cosines after all. Namely, take  $r_1 = \alpha + \beta i$ ,  $r_2 = \alpha - \beta i$  the two roots of the characteristic polynomial, and look at

$$C_1 e^{(\alpha+\beta i)t} + C_2 e^{(\alpha-\beta i)t}$$

Does it make any sense to say this is the general solution? Not yet, but we can massage it to get real solutions out of it: expanding out the exponential gives

$$\begin{aligned} & C_1 e^{\alpha t} e^{\beta i t} + C_2 e^{\alpha t} e^{-\beta i t} = \\ & e^{\alpha t} (C_1 \cos(\beta t) + i C_1 \sin(\beta t) + C_2 \cos(\beta t) - i C_2 \sin(\beta t)). \end{aligned}$$

To perform the last step we used Euler's formula with  $\beta t$  in place of  $t$ :  $e^{i(\beta t)} = \cos(\beta t) + i \sin(\beta t)$  together with the fact that cosine is even and sine is odd. The final trick is to allow  $C_1, C_2$  to themselves be complex numbers if necessary (why not?) but choose them carefully so as to get *real solutions*. For instance, if we take  $C_1 = C_2 = 1/2$ , we get  $x_1(t) = e^{\alpha t} \cos(\beta t)$ , whereas if we take  $C_1 = -i/2$ ,  $C_2 = i/2$ , we get  $x_2(t) = e^{\alpha t} \sin(\beta t)$ . Thus we have found two different real solutions, and the general solution must be

$$C_1 e^{\alpha t} \cos(\beta t) + C_2 e^{\alpha t} \sin(\beta t), r_1 = \alpha + \beta i, r_2 = \alpha - \beta i$$

**5.2. Motion of a spring.** We can illustrate our analysis of this second-order equation via the motion of a mass on a spring. Assume that we have a mass  $m$  attached to a spring, and assume first that friction does not play a role. Then Newton's second law tells us

$$F = ma = mx''$$

and Hooke's law tells us

$$F(x) = -kx$$

where  $k > 0$  is a constant measuring the strength of the spring; the minus sign is there because the force imparted by the spring is always opposite to the direction of motion. We get

$$x'' + k/mx = 0$$

which has characteristic polynomial  $r^2 + k/m = 0$ ; the roots are  $\pm\sqrt{k/m}i$ , so that  $\alpha = 0, \beta = \sqrt{k/m}$  and the general solution is  $C_1 \cos(\sqrt{k/m}t) + C_2 \sin(\sqrt{k/m}t)$ . The general solution is visibly periodic, with period  $\frac{2\pi}{\sqrt{k/m}}$ . To see that it is actually a "sinusoidal function" we should recall the trigonometric identity

$$A \cos t + B \sin t = \sqrt{A^2 + B^2} \cos(t - \theta), \theta = \arctan(B/A)$$

(This is, perhaps, the least-remembered of all the standard trig. identities; one can derive it using  $\cos(\alpha - \beta) = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)$ .) Applying it in our context we get

$$x(t) = \sqrt{C_1^2 + C_2^2} \cos(\sqrt{k/m}t - \theta), \theta = \arctan(C_2/C_1)$$

which shows that the general solution is a cosine function. (Similarly, one can show it is a sine function; hence it is sinusoidal.)

In fact the notion of a spring which oscillates forever with no loss of energy is not physically realistic (it would be a "perpetual motion machine"). In the real world, the spring will lose energy due to friction. We shall model the frictional force as being proportional to the velocity but in the opposite direction (is this really the truth? ask a physicist), so that the revised version of Hooke's law reads

$$F(x) = -kx - fx'$$

where  $f > 0$  is another proportionality constant measuring the magnitude of the friction and the minus sign is there for the same reason as before; setting  $F = ma$  we get

$$x'' + f/mx' + k/mx = 0$$

And now something interesting occurs. We would imagine that, due to loss of energy, all solutions to this differential equation would satisfy  $\lim_{t \rightarrow \infty} x(t) = 0$  and this turns out to be correct. One might also imagine (I would, anyway) that we would experience a damped oscillation: i.e. the solutions would cross equilibrium infinitely many times but each time not make it as far from the equilibrium point. But this will only happen if the characteristic polynomial has complex roots: when  $D = (f/m)^2 - 4k/m < 0$ , i.e. when the frictional force is relatively small compared to the spring constant. In this case the roots are  $(-f/m \pm \sqrt{|(f/m)^2 - 4k/m}|i)/2$ , so  $a = -f/2m < 0$  and  $b = \sqrt{|D|}/2$  and we get

$$x(t) = e^{at}(C_1 \cos(\beta t) + C_2 \sin(\beta t))$$

Since  $\alpha < 0$ ,  $e^{\alpha t}$  is exponentially decaying at infinity; the oscillatory term is bounded by  $\sqrt{C_1^2 + C_2^2}$ , so the solution goes to zero but returns to equilibrium infinitely many times. This is called the *underdamped case*.

Suppose now  $D = 0$ , i.e. we have a repeated real root of  $r = -f/2m < 0$ . Then the general solution is

$$x(t) = e^{rt}(C_1 + C_2t)$$

On one hand,  $r < 0$  and the exponential decay beats the polynomial growth (apply L'Hopital's rule) and we get  $\lim_{t \rightarrow \infty} x(t) = 0$ . On the other hand, the solutions are no longer oscillatory:  $x(t) = 0$  exactly when  $C_1 + C_2t = 0$ , i.e. at the unique time  $t = -C_1/C_2$  (which could be negative, so that if we imagine setting up our system at time 0 we would never see a return to equilibrium). This is called the *critically damped case*.

Finally, suppose  $D > 0$ , and we have distinct real roots  $r_1, r_2 = -f/2m \pm \sqrt{(f/m)^2 - 4k/m}/2$ . Notice that both of these roots are negative, indeed, the larger one is  $-f/2 + 1/2\sqrt{(f/m)^2 - 4k/m} < -f/2 + 1/2\sqrt{(f/m)^2} = 0$ , so the larger one (hence also the smaller one) is negative. The general solution is

$$x(t) = C_1e^{r_1t} + C_2e^{r_2t}$$

the sum of two exponentially decaying functions, hence certainly goes to zero at infinity. This time the solution may *never* cross the equilibrium point:

$$0 = x(t) = e^{r_1t}[C_1 + C_2e^{(r_2-r_1)t}]$$

so we have to solve  $e^{(r_2-r_1)t} = -C_1/C_2$ . Notice that the exponential of anything is positive, so the right-hand side must be positive – i.e.  $C_1$  and  $C_2$  must have opposite signs – if there is to be any solution at all. Assuming this, we take logs to get  $(r_2 - r_1)t = \ln(-C_1/C_2)$ , or  $t = \frac{\ln(-C_1/C_2)}{r_2 - r_1}$  is the unique time value of a solution (which again may be positive or negative). It follows that in this, the *overdamped* case, there will be either one or no returns to equilibrium, depending on the coefficients  $C_1, C_2$  – i.e. depending on the initial conditions. Recall that on your last problem set you explored how to choose  $C_1, C_2$  so that various behaviors occur.

**Problem 3:** Suppose  $y(t)$  is a solution to  $y'' + by' + cy = 0$  which is bounded for all  $t$  (that is, there is a constant  $M$  such that  $|y(t)| \leq M$  for all  $t$ ) and such that  $\lim_{t \rightarrow \infty} y(t) = 0$ . What can you say about  $y$ ?

**Problem 4:** Suppose that  $y(t)$  is a solution to  $y'' + by' + cy = 0$  such that  $y(t) = 0$  precisely when  $t$  is an integer ( $\dots, -2, -1, 0, 1, 2, \dots$ ) and such that  $y'(0) = 5$ .

- (a) Does this uniquely determine  $y(t)$ ? What is the equation?  
 (b) If your answer to part (b) was no (hmm...), what if we add the condition that the maximum value of  $y$  is 5 for all values of  $t$ , both positive and negative? Does this uniquely determine  $y$ ?

**Problem 5:** Superman is battling his archnemesis Doomsday. Superman happens upon a gigantic *overdamped* spring.

- a) Suppose Superman holds the spring with the mass stretched to  $P$  units to the left of the equilibrium position. Doomsday is standing at the equilibrium position.

Can Superman hit Doomsday with the mass on the spring?

b) Suppose now that Lois Lane stands at  $P/2$  units to the left of the equilibrium position, i.e., between Superman and Doomsday. Obviously the strategy of the previous problem is not a good one. Superman decides instead to hurl the mass *to the left* so that he has time to rescue Lois before the spring doubles back and (he hopes) hits Doomsday. Will his strategy succeed?

Hint: The point of it being Superman is that he can impart an arbitrarily large initial velocity; in the first case it is a large positive velocity; in the second case it is a large negative velocity.

## 6. POWER SERIES SOLUTIONS

There are many differential equations that cannot be solved exactly, just as there are many functions for which antiderivatives cannot be explicitly found. In the latter situation we used power series to represent these antiderivatives; we can do something very similar to solve differential equations; namely we postulate a power series solution  $y(t) = \sum_{n=0}^{\infty} a_n t^n$  and we substitute this into the differential equation. We will then be able to solve for the coefficients in terms of the first few coefficients. More precisely, if it is a first-order differential equation, then  $a_0 = x(0)$  gives us our initial condition and remains arbitrary, and we can solve for the remaining coefficients in terms of  $a_0$ . If it is a second-order differential equation then  $a_0 = x(0), a_1 = x'(0)$  give the initial conditions and remain arbitrary, and we can solve for the remaining coefficients in terms of  $a_0, a_1$ . In general, we have to be somewhat fortunate in recognizing a pattern for every power series coefficient; we may well have to content ourselves with writing down the coefficients up to some order, like up to  $t^3$  or  $t^4$ .

Example: Consider our old friend the differential equation  $dy/dt = y$ . Suppose  $y(t) = \sum a_n t^n$  is a power series solution. Then

$$y(t) = a_0 + a_1 t + a_2 t^2 + \dots + a_{n-1} t^{n-1} + \dots$$

$$dy/dt = a_1 + 2a_2 t + 3a_3 t^2 + \dots + n a_n t^{n-1} + \dots$$

Our equation is  $y = dy/dt$ ; recall that the only way two power series can be equal as functions is if all their coefficients are equal; that is we are setting  $a_1 = a_0, 2a_2 = a_1, \dots, n a_n = a_{n-1}$ . Rewriting the general equality as  $a_n = a_{n-1}/n$ , it is telling us that we get from one term to the next by dividing by  $n$ . So  $a_1 = a_0, a_2 = a_1/2 = a_0/2, a_3 = a_2/3 = a_0/3!, a_4 = a_3/4 = a_0/4!$ , and in general we get from  $a_0$  to  $a_n$  by dividing by all the numbers between 1 and  $n$ , so  $a_n = a_0/n!$ . Thus,  $y(t) = \sum a_0/n! t^n = a_0 \sum_{n=0}^{\infty} t^n/n!$ . This last term is  $a_0$  times the Taylor expansion (at  $t = 0$ ) of the exponential function  $e^t$ ; we also showed (using the theorem we love to hate, Taylor's remainder theorem) that  $e^t$  is equal to its Taylor series, so on the one hand we already know that we have gotten  $y(t) = a_0 e^t$  as the general solution to the equation  $dy/dt = y$ , which is correct. On the other hand, suppose that we did not know that  $e^t = \sum t^n/n!$ . Consider functions  $y_1(t) = e^t$  and  $y_2(t) = \sum_{n=0}^{\infty} t^n/n!$ . Both of these functions satisfy the initial value problem

$$dy/dt = y, y(0) = 1$$

and by Theorem 1, initial value problems have unique solutions. It must be that  $y_1 = y_2$ , i.e.  $e^t = \sum_{n=0}^{\infty} t^n/n!$  That is, if we can show that a function and its Taylor series satisfy the same initial value problem, they must be equal, whenever both are defined. This is actually quite a versatile technique for showing that functions are equal to their Taylor expansions.

Example: Consider our friend the second-order differential equation  $y'' = -y$ . Suppose  $y(t) = \sum a_n t^n$  is a power series solution. Then

$$y(t) = a_0 + a_1 t + a_2 t^2 + \dots + a_{n-1} t^{n-1} + \dots$$

$$dy/dt = a_1 + 2a_2 t + 3a_3 t^2 + \dots + n a_n t^{n-1} + \dots$$

$$-y'' = -2a_2 - 3 \cdot 2a_3 t - 4 \cdot 3a_4 t^2 - \dots - n(n-1)a_n t^{n-2} - \dots$$

Let us group the even- and odd-numbered terms separately. For the evens we get

$$-2a_2 = a_0, -4(3)a_4 = a_2, -6(5)a_6 = a_4, \dots$$

and solving this we get  $a_2 = (-1)a_0/2!$ ,  $a_4 = (-1)^2 a_0/4!$ ,  $\dots$ ,  $a_{2n} = (-1)^n a_0/(2n)!$  Whereas for the odds we get

$$-3(2)a_3 = a_1, -5(4)a_5 = a_3, -7(6)a_7 = a_5, \dots$$

and we get  $a_3 = (-1)a_1/3!$ ,  $a_5 = (-1)a_1/5!$ ,  $\dots$ ,  $a_{2n+1} = (-1)^n a_1/(2n+1)!$  Putting everything together we get the general solution

$$x(t) = a_0 \sum_{n=0}^{\infty} (-1)^n t^{2n}/(2n)! + a_1 \sum_{n=0}^{\infty} (-1)^n t^{2n+1}/(2n+1)!$$

which again we recognize as  $a_0 \cos t + a_1 \sin t$ , or at least the Taylor series at 0 thereof. But as above, we can rederive the equality of sine and cosine with their Taylor series: both  $y_1(t) = \cos t$  and  $y_2(t) = \sum_{n=0}^{\infty} (-1)^n t^{2n}/(2n)!$  solve the second-order initial value problem

$$y'' = -y, y(0) = 1, y'(0) = 0$$

so must be equal, and both  $y_3(t) = \sin t$  and  $y_4(t) = \sum_{n=0}^{\infty} (-1)^n t^{2n+1}/(2n+1)!$  solve the initial value problem

$$y'' = -y, y(0) = 0, y'(0) = 1$$

so they must be equal.

## 7. MODELLING WITH DIFFERENTIAL EQUATIONS

Differential equations are widely used to model various real-world phenomena: a certain *law* is given describing the rate of change of a certain quantity in terms of the quantity of itself; the *solution* to the differential equation tells us how the quantity changes over time under that law. In our course, one of the things we ask you to do is to, given a law expressed in words, write down the corresponding differential equation. Here are some common instances of this:

**7.1. Exponential growth and decay.** If the rate of change of a quantity is proportional to the current amount of that quantity, that dictates the differential equation  $dy/dt = ky$ , where  $k$  is a proportionality constant. If  $k > 0$  this is called exponential growth; if  $k < 0$  this is called exponential decay; notice that the general solution to this equation is  $y(t) = y_0 e^{kt}$ . This type of differential equation models many things, including: unrestricted population growth, compound interest and loans, radioactive decay.

**Problem 6:** You invest \$1000 in the bank. The interest is compounded continuously (N.B.: this means it is modelled by a differential equation  $dy/dt = ky$ ). After 12 years your money has doubled. What was the interest rate?

**Problem 7:** (doubling time): Suppose you invest money at an interest rate of  $k\%$ . Give an expression in terms of  $k$  for the amount of time it takes for your money to double, and show that this is independent of the size of your original investment. Why is this called the law of 72?

**7.2. rate = rate<sub>in</sub> - rate<sub>out</sub>.** In many of the problems in which we ask you to write down a differential equation, there will be one or more positive rates of change and one or more negative rates of change. The key is that the total rate of change is obtained by adding up all the positive rates of change and subtracting the negative rates of change; or rate = rate in - rate out.

Example: A nonfatal disease is sweeping through the small town of Springfield at a rate which is proportional to the current population of Springfield. (Insert a funny, Simpsons related disease here) To combat the disease the mayor and the police chief have decided that all infected will be sent out of the town. However, the only way out of town is via a single bridge, so that only 5000 people per day can leave town. Write a differential equation that models the spread of the disease. Is Quimby and Wiggum's idea a good one?

The rate in is  $(dy/dt)_{in} = ky$ , where  $k$  is some proportionality constant. The rate out is  $(dy/dt)_{out} = 5000$ . So the total rate of change is  $dy/dt = ky - 5000$ . As for whether this is a good idea: notice that this is an autonomous equation; the unique equilibrium solution is at  $y = 5000/k$ ; above this the solutions increase to  $\infty$  and below this the solutions decrease (to  $-\infty$  actually, but we'll cut off at zero). Thus, if the initial infected population is less than  $5000/k$ , then eventually the disease will be eradicated from Springfield (at a cost of sending away many of its citizens, of course), whereas if the initial infected population is  $> 5000/k$ , then eventually the entire town will become infected.

Example: Newton's law of cooling states that if we bring an object into a room, the rate of change of the object's temperature is proportional to the difference between the object's temperature and the ambient temperature. Write a differential equation modelling this.

Solution : Let  $y(t)$  denote the object's temperature at time  $t$  and let  $T$  be the

(fixed) ambient temperature. Newton's law seems to be telling us that

$$dy/dt = k(y - T)$$

but we need to think about what's going on. If  $k$  were positive (as proportionality constants are usually taken to be), then the sign analysis of this (autonomous!) equation would tell us that the temperature is increasing when the initial temperature is greater than the ambient temperature and decreasing when the initial temperature is less than the ambient temperature; this is exactly the opposite of what should happen. Therefore we are missing a minus sign: assuming  $k > 0$  is a positive constant, the equation should be

$$dy/dt = k(T - y).$$

**Problem 8:** (mixture problems): We have yellow and blue paint mixing in an enormous paint-mixer which is simultaneously spraying out green paint. The mixture replenishes the paint at a fixed proportion of  $y\%$  yellow paint and  $(1 - y)\%$  blue paint. The paint enters the mixer at a rate of  $r_I(t)$  gallons per second and leaves the mixer at a rate of  $r_O(t)$  gallons per second; we assume that the mixer mixes the paint instantly. Assume that the mixer is initially filled with 1000 gallons of yellow paint (it really is enormous). Write differential equations for  $da/dt, db/dt$  the rates of change of the yellow paint and the blue paint respectively. Don't worry about what happens when there is a negative amount of one of the paints.

(Hint: use rate in minus rate out. The rate out for, say, the yellow paint is going to be equal to the concentration of the yellow paint times the total rate out. The concentration is in turn the current quantity of yellow paint divided by the current total quantity of paint, i.e. the yellow paint + the blue paint.)

## 8. SOLUTIONS

Solution 1: (1a) The  $5y$  and the  $-5y$  cancel, and this is just the integration equation  $dy/dt = te^t$ . Integrating by parts, the general solution is  $y = te^t - e^t + C$ .

(1b) This is an autonomous equation. The equilibrium solutions are at the zeroes of the cosine function, i.e. at  $\pi/2 + n\pi$  for integers  $n$ . In particular there are infinitely many strips and all of them are bounded. The sign of the cosine function alternates between positive and negative, so the solutions alternate between being increasing and decreasing; the one with  $y(0) = 0$  is increasing.

(1c) This is a linear equation with  $p(t) = -1$ ,  $-q(t) = e^t$ , so  $P = -t$  and the general solution is  $y(t) = \frac{-\int e^{-t}e^t dt}{e^{-t}} + Ce^t = te^t + Ce^t$ .

(1d) This is an autonomous equation which is similar to (3b) – there are infinitely many equilibrium solutions corresponding to the values where the sine function is 1 – that is  $\pi/2 + 2n\pi$  for integers  $n$ . But this time 0 is the minimum value of the function  $\sin y + 1$ , so the solutions are increasing in every strip.

(1e) This is a separable equation; separating variables we get  $\int \cos y dy = 1/4 \int \frac{dt}{t^2}$ ;

integrating both sides gives  $\sin y = \frac{-1}{4t} + C$ .

Solution 2:

- (a) i) False ii) True (iii) True
- (b) The nullclines are the  $j$  and  $r$  axes. For  $j > 0$  orient the  $j$  axis to the right and for  $j < 0$  orient the  $j$  axis to the left. For  $r > 0$  orient the  $r$  axis down and for  $r < 0$  orient the  $r$  axis up. That means that the first quadrant is right and down, the second is right and up, the third is left and up and the fourth is down and left. The only equilibrium point is  $(0, 0)$ .
- (c) The solution trajectories are concentric circles centered about the origin and oriented clockwise.
- (d) We're starting with Romeo in love with Juliet and Juliet non-committal. Juliet will start to dislike Romeo ( $j$  becomes negative) and the because she dislikes him, Romeo's ardor starts to wane ( $r$  becomes less positive.) By the time  $j = -1$  (Juliet's distaste for him is strong), Romeo's feelings are neutral. (They're at the point  $(0, -1)$ .) They travel around the circle  $r^2 + j^2 = 1$  on a never-ending cycle of love and loathing. (Nobody dies in Strogatz's version, but it is a tragedy anyway.)
- (e)  $r(t) = \cos t$  and  $j(t) = -\sin t$ .

Solution 3: In fact  $y$  is identically zero. The idea is that if there are any exponentials at all in the solution – i.e. unless the roots of the characteristic polynomial are purely imaginary – then every solution will become arbitrarily large in magnitude either for very large positive time (if it is an exponential to a positive power) or for very large negative time (if it is an exponential to a negative power). So the only bounded solutions are the purely periodic ones. But then a periodic function can't tend to zero unless it is identically zero.

Solution 4: In part (a), we need some kind of oscillation with a period of 1, so the characteristic polynomial needs to have complex roots  $a \pm bi$  with  $b = 2\pi$ . But the real part of  $a$  could be arbitrary: if  $a < 0$  we will see an (under)damped oscillation (underdamped spring); if  $a = 0$  we will get purely periodic solutions (undamped spring); and if  $a > 0$  we will get oscillatory solutions whose amplitude tends to  $\infty$  (not interpretable via our spring model). So the answer is that  $y(t)$  is not uniquely determined. Passing to part (b), the only way we get solutions which are bounded for all time (see Problem 4) is if they are periodic – so that we need  $a = 0$ , i.e. we are looking for functions of the form  $y(t) = C_1 \cos(2\pi t) + C_2 \sin(2\pi t)$ . The conditions that  $y(0) = 0$  and maximum value 5 mean that  $C_1 = 0$  and the amplitude is 5, so  $y(t) = 5 \sin(2\pi t)$ .

Solution 5: We know the general solution is  $C_1 e^{r_1 t} + C_2 e^{r_2 t}$ , with  $r_1, r_2$  both negative. Say we have labeled the roots such that  $r_1 < r_2$  (they're distinct, so one of them has to be less!) The initial conditions are

$$x(0) = C_1 + C_2 = -P, v(0) = r_1 C_1 - r_2 C_2$$

Solving for  $C_2 = -C_1 - P$ , we get  $v(0) = r_1 C_1 + r_2(-C_1 - P) = (r_1 - r_2)C_1 - r_2 P$ , and the equation is

$$C_1 e^{r_1 t} + (-P - C_1) e^{r_2 t}$$

For part a),  $v(0)$  is to be very large and positive; as it approaches infinity we must have that  $C_1$  is itself very large and positive. So now we return to the value of time for which the spring intersects the equilibrium:  $t = \frac{\ln(\frac{C_1}{P+C_1})}{r_2-r_1}$ . First we must verify that we are taking the log of something positive, and this is true; since  $C_1$  is positive, the numerator and the denominator are both positive. Next, we need the unique time value to be *positive*. We are taking the log of a number which is less than 1, hence negative, but also the denominator is negative, so the entire fraction is positive. Therefore the answer to part a) is that Superman hits Doomsday. (At least, the spring reaches the equilibrium point. But as  $C_1$  approaches infinity,  $t$  approaches zero, so if he throws it fast enough Doomsday will not be able to get out of the way in time.) For part b), the equations are the same, but now  $C_1$  is very negative, and if  $C_1$  is large and negative, then  $C_1 + P$  will also be negative but will be smaller in absolute value than  $C_1$ , i.e. the quantity  $\frac{C_1}{P+C_1} > 1$ . So in the expression for  $t$  the numerator is positive and the denominator is negative, so the entire fraction is negative, meaning that no matter how hard Superman flings the mass to the left, it will never return to the equilibrium point and hit Doomsday – even Superman can't make an overdamped spring return to the equilibrium position in this way.

Solution 6: The general solution to the differential equation is  $y(t) = y_0 e^{kt}$ . We are told that  $2y_0 = y(12) = y_0 e^{12k}$ , or  $k = \frac{\ln 2}{12}$ , or approximately 5.8%. // // Solution 7: This is much like the previous problem:  $2y_0 = y_0 e^{kt}$ , so  $e^{kt} = 2$  and  $t = \frac{\ln 2}{k}$ , which is approximately  $\frac{.693}{k}$ . As for why it's called the law of 72 – this is what Pete's father called it when he explained it to him as a child – one can only guess that 72 is pretty close to 69.3; that 72 has more factors than does 69 – so that if the interest rate is say, 8% it will take about 9 years for your money to double (and in fact a little less, so this is a prudently conservative estimate), and that his father did not want to tell him about “the law of 69”.

Solution 8: The total rate of change of the amount of paint in the mixer is  $r_I - r_O$ , so the total amount of paint in the mixer at time  $t$  is  $R_I(t) - R_O(t)$ , where  $R_I' = r_I$ ,  $R_O' = r_O$  and the antiderivatives are chosen so that  $R_I(0) - R_O(0) = 1000$ . Let us now consider the yellow and the blue paint separately. As for the yellow paint,  $y\%$  of the incoming paint is yellow, so the rate in for the yellow paint is  $(y\%) \cdot r_I(t)$ . The rate out for the yellow paint is equal to the total rate out times the concentration of the yellow paint at time  $t$ ; the concentration is the current amount of yellow paint,  $a(t)$ , divided by the total amount of paint,  $R_I(t) - R_O(t)$ , or  $\frac{a(t)}{R_I(t) - R_O(t)}$ , and the rate out for the yellow paint is  $\frac{a(t)}{R_I(t) - R_O(t)} \cdot r_O(t)$ . Therefore the total rate of change for the yellow paint is

$$da/dt = (y\%) \cdot r_I(t) - \frac{a(t)}{R_I(t) - R_O(t)} \cdot r_O(t).$$

Running through the same argument for the blue paint we get

$$db/dt = (1 - y\%) \cdot r_I(t) - \frac{b(t)}{R_I(t) - R_O(t)} \cdot r_O(t).$$

Observe a couple of things: first, since  $R_I(t) - R_O(t) = a(t) - b(t)$ , indeed  $da/dt + db/dt = r_I(t) - r_O(t)$  gives the total rate of change; this is a check that our answer

is reasonable. Second, notice that this is *not* essentially a system of differential equations; it is two independent first-order equations. Moreover each equation is a *linear* first-order equation, so that we could actually explicitly solve for  $a(t)$  and  $b(t)$ .