Math 1b Midterm I Solutions
Tuesday, March 14, 2006

March 15, 2006

1. (6 points) Which of the following gives the area bounded on the left by the y-axis, on the right by the curve $y = 3 \arcsin x$ and above by $y = 3\pi/2$? Please circle ***ALL*** correct answers. No written justification is required.

   (a) $\int_0^1 (3\pi/2 - 3 \arcsin x)\, dx$

   (b) $\int_0^{3\pi/2} (3\pi/2 - 3 \arcsin x)\, dx$

   (c) $\int_0^{3\pi/2} \sin(\frac{y}{3})\, dy$

   (d) $\int_0^1 \sin \frac{y}{3}\, dy$

   (e) $\frac{1}{3} \int_0^{3\pi/2} \sin y\, dy$

   Solution. (a) and (c)

2. (11 points)

   (a) Let $f(x) = 3x^2 - 2x + k$. Find k such that the average value of f on the interval $[-1, 2]$ is 3.

   (b) Suppose that f is a continuous function. If the average value of f over the interval $[-3, 1]$ is 2 and the average value of f over the interval $[1, 7]$ is 5, what is the average value of the function over the interval $[-3, 7]$? ***Hint***: The answer is not 7/2.

   Solution.

   (a) The average value of f is

$$\begin{aligned} 3 &= \frac{1}{2-(-1)} \int_{-1}^{2} f(x)\, dx \\ &= \frac{1}{3} \int_{-1}^{2} 3x^2 - 2x + k\, dx \\ &= \frac{1}{3}(x^3 - x^2 + kx)\Big|_{-1}^{2} = 2 + k. \end{aligned}$$

Thus, $k = 1$.

(b) Since the average value of f on $[-3, 7]$ will be the same as the average value of the function

$$g(x) = \begin{cases} 2, & \text{for } -3 \leq x < 1, \\ 5, & \text{for } 1 \leq x \leq 7, \end{cases}$$

we know that

$$\frac{1}{10} \int_{-3}^{7} g(x)\, dx = \frac{1}{10} \int_{-3}^{1} 2\, dx + \frac{1}{10} \int_{1}^{7} 5\, dx = \frac{38}{10}.$$

is the average value of f

3. (15 points) Consider the region bounded curves

$$\begin{aligned} y &= x^2 - 1 \\ y &= -x + 1 \end{aligned}$$

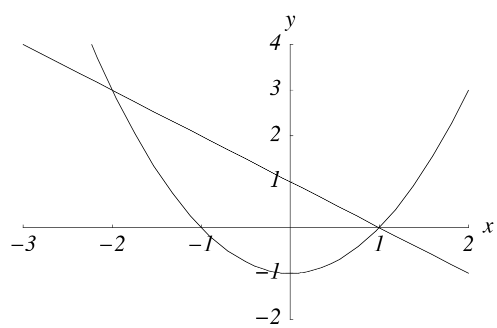
(a) Sketch the area bounded by these curves.

(b) Calculate the area between both of these curves.

(c) Find the volume of the solid of revolution obtained by rotating the region enclosed by the two curves around the horizontal line $y = -1$.

Solution.

(a)



(b) The curves $y = x^2 - 1$ and $y = -x + 1$ intersection at $x = -2$ and $x = 1$. Thus, the area between the two curves is

$$\int_{-2}^{1} (-x + 1) - (x^2 - 1)\, dx = \int_{-2}^{1} -x^2 - x + 2\, dx = \frac{9}{2}.$$

(c) The volume is given by

$$\int_{-2}^{1} \pi[(-x + 1) + 1]^2 - \pi[(x^2 - 1) + 1]^2\, dx = \pi \int_{-2}^{1} (-x + 2)^2 - x^4\, dx = \frac{72}{5}\pi.$$

4. (11 points) The soot produced by a garbage incinerator spreads out in a circular pattern. The depth of soot that is currently on the ground is $H(r) = e^{-2r}/10$ millimeters deep, where r is the number of kilometers from the incinerator. What is the total volume of soot that has been deposited within a 5 km radius of the incinerator? Remember that 1000 mm is one meter and 1000 m is one kilometer.

Solution. If we change all of the units to kilometers, the depth of the soot at a distance r from the garbage incinerator is $H(r) = 10^{-7}e^{-2r}$ km. If we divide the area into circular slices, then the area of a single slice is approximately $2\pi r\,\Delta r$. Thus, the amount of soot inside the area can be approximated by the Riemann sum

$$\sum_{i=1}^{n} 2\pi r H(r)\,\Delta r.$$

Letting $n \to \infty$, we obtain

$$\begin{aligned}
\int_0^5 2\pi r H(r)\,dr &= 2 \cdot 10^{-7} \cdot \pi \int_0^5 r e^{-2r}\,dr \\
&= 2 \cdot 10^{-7} \cdot \pi \left[-\frac{r}{2} e^{-2r} - \frac{1}{4} e^{-2r} \right]_0^5 \\
&= 2 \cdot 10^{-7} \cdot \pi \left[-\frac{5}{2} e^{-10} - \frac{1}{4} e^{-10} + \frac{1}{4} \right] \\
&= \left(-5e^{-10} - \frac{1}{2} e^{-10} + \frac{1}{2} \right) \cdot 10^{-7}\pi \\
&\approx 1.57 \times 10^{-7} \text{ km}^3 \text{ or } 157 \text{ m}^3
\end{aligned}$$

5. (11 points) A snail crawls along the curve $y = \sqrt{x^3}$ at a speed of 3 ft/hr. How long does it take the snail to crawl from the point $(1,1)$ to $(4,8)$, where the x and y-coordinates are given in feet.

Solution. The length of the path traveled by the snail is

$$\begin{aligned}
\int_1^4 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx &= \int_1^4 \sqrt{1 + \left(\frac{3}{2}\sqrt{x}\right)^2}\,dx \\
&= \int_1^4 \sqrt{1 + \frac{9}{4}x}\,dx \\
&= \left[\frac{8}{27} \left(1 + \frac{9}{4}x\right)^{3/2} \right]_1^4 \\
&= \frac{1}{27}(80\sqrt{10} - 13\sqrt{13})
\end{aligned}$$

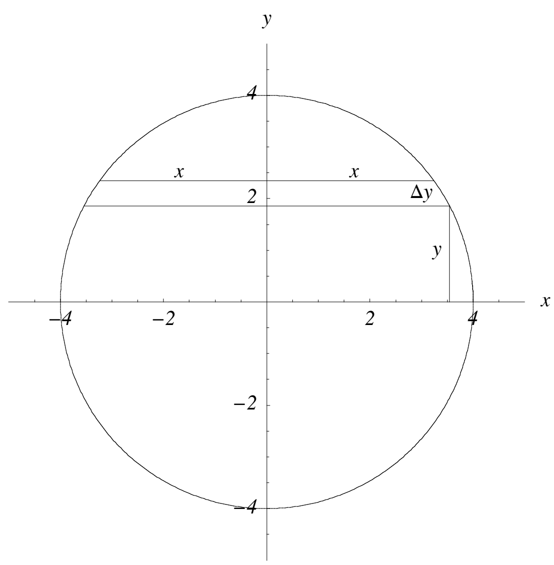
At a speed of 3 ft/hr, the snail will cover the distance in

$$\frac{1}{81}(80\sqrt{10} - 13\sqrt{13}) \approx 2.54 \text{ hours.}$$

6. (11 points) A cylindrical gasoline tank with radius 4 ft and length 15 ft is buried under a service station. The top of the tank is 10 ft underground, and its flat ends are perpendicular to the ground's surface. Find a definite integral that will tells us the total amount of work needed to pump all of the gasoline in the tank to a nozzle that is 3 ft above the ground. (Gasoline weighs $\rho = 42$ lb/ft^3.) ***You do not need to evaluate the integral.***

Solution. If we place the center of circular side of the tank at the origin, then the volume of a rectangular slice of the tank is

$$15 \cdot 2x \, \Delta y = 30x \, \Delta y = 30\sqrt{16 - y^2} \, \Delta y.$$



Since the slice must be lifted $17 - y$ ft, the amount of work required to list one slice is

$$W_i \approx 42 \cdot 30(17 - y)\sqrt{16 - y^2} \, \Delta y.$$

Thus, the total amount of work required to drain the tank is

$$W = \int_{-4}^{4} 1260(17 - y)\sqrt{16 - y^2} \, dy.$$

Other solutions are possible depending on how one chooses the coordinate system.

7. (11 points) The base of a solid object is the region bounded by $y = 1/x$, $y = 0$, $x = 1$, and $x = 4$. Every cross-section of the solid taken perpendicular to the x-axis is a square. What is the volume of the object?

Solution. The area is

$$\int_1^4 A(x) \, dx = \int_1^4 \frac{1}{x^2} \, dx = \frac{3}{4}.$$

8. (11 points) Hoping to estimate the volume of wood in a 20-meter log, Ranger Smith uses a tape measure to gauge the log's girth (circumference) at 5 meter intervals, starting from the large end of the log. Here are Ranger Smith's results.

Measuring a Natural Log					
distance from the end (m)	0	5	10	15	20
circumference (m)	4.0	3.6	3.2	2.6	2.0

(a) Let x be the distance in meters from the large end of the log and $c(x)$ be the girth of the log at x. Write a definite integral that approximates the volume of the log.

(b) Write out an arithmetic expression that uses the Trapezoid Rule to estimate the volume of the log for $n = 4$. ***You do not need to evaluate the expression.***

(c) Write out an arithmetic expression that uses the Simpson's Rule to estimate the volume of the log for $n = 4$. ***You do not need to evaluate the expression.***

Solution.

(a) $\displaystyle\int_0^{20} \pi \left(\frac{c(x)}{2\pi}\right)^2 dx = \int_0^{20} \frac{[c(x)]^2}{4\pi}\, dx$

(b) $\displaystyle\frac{5}{2}\left(\frac{(4.0)^2}{4\pi} + 2\cdot\frac{(3.6)^2}{4\pi} + 2\cdot\frac{(3.2)^2}{4\pi} + 2\cdot\frac{(2.6)^2}{4\pi} + \frac{(2.0)^2}{4\pi}\right)$

(c) $\displaystyle\frac{5}{3}\left(\frac{(4.0)^2}{4\pi} + 4\cdot\frac{(3.6)^2}{4\pi} + 2\cdot\frac{(3.2)^2}{4\pi} + 4\cdot\frac{(2.6)^2}{4\pi} + \frac{(2.0)^2}{4\pi}\right)$

9. (13 points)

(a) Find $\displaystyle\int_0^{\infty} \frac{1}{1+x^2}\, dx$.

(b) Determine whether

$$\int_1^{\infty} \frac{2}{1+x^5}\, dx$$

converges. Explain your reasoning completely.

(c) Determine whether

$$\int_{-\infty}^{\infty} \sin 2x\, dx$$

converges or diverges.

Solution.

(a) $\displaystyle\int_0^{\infty} \frac{1}{1+x^2}\, dx = \arctan x\Big|_0^{\infty} = \frac{\pi}{2}$

(b) The improper integral

$$\int_1^\infty \frac{2}{1+x^5}\,dx$$

converges since

$$0 < \frac{2}{1+x^5} \leq \frac{2}{1+x^2}$$

and

$$\int_1^\infty \frac{2}{1+x^2}\,dx = 2\arctan x\Big|_1^\infty = \frac{\pi}{2}.$$

(c) The integral

$$\int_{-\infty}^\infty \sin 2x\,dx$$

diverges since

$$\int_0^\infty \sin 2x\,dx = -\frac{\cos 2x}{2}\Big|_0^\infty$$

and $\lim_{x\to\infty} \cos 2x$ does not exist.