Solutions to Math 1b Midterm II

Tuesday, April 18, 2006

1. (6 points) Suppose that the power series $\sum_{n=0}^{\infty} a_n(x+2)^n$ converges if $x = -7$ and diverges if $x = 7$. Decide which of the following series must converge, must diverge, or may either converge or diverge (inconclusive). Circle your answer. *You do not need to justify your answers.*

   (a) If $x = -8$, the power series *Converges* *Diverges* *Inconclusive*

   (b) If $x = 1$, the power series *Converges* *Diverges* *Inconclusive*

   (c) If $x = 3$, the power series *Converges* *Diverges* *Inconclusive*

   (d) If $x = -11$, the power series *Converges* *Diverges* *Inconclusive*

   (e) If $x = 5$, the power series *Converges* *Diverges* *Inconclusive*

   (f) If $x = -5$, the power series *Converges* *Diverges* *Inconclusive*

   Solution. (a) *Inconclusive*, (b) *Converges*, (c) *Inconclusive*, (d) *Inconclusive*, (e) *Inconclusive*, (f) *Converges.*

2. (20 points) Determine whether each series below converges absolutely, converges conditionally, or diverges. Be careful to justify each of your answers by explicitly referring to the test that you used and explaining how you used it.

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^4+n+1}$

   (b) $\displaystyle\sum_{n=0}^{\infty} \left(\frac{1}{2^n}+\frac{1}{3^n}\right)^2$

   (c) $\displaystyle\sum_{n=1}^{\infty} \frac{\sin(1/n)}{n^3}$

   (d) $\displaystyle\sum_{n=2}^{\infty} \frac{2^n}{n!(\ln n)^3}$

(e) $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln n}$

Solution.

(a) Since
$$\sum_{n=1}^{\infty} \frac{1}{n^4+n+1} < \sum_{n=1}^{\infty} \frac{1}{n^4},$$
and the series on the right-hand side of the inequality is a convergent p-series, the series converges by the Comparison Test. Since all of the terms are positive, the series converges absolutely.

(b) Since
$$\sum_{n=0}^{\infty} \left(\frac{1}{2^n} + \frac{1}{3^n}\right)^2 < \sum_{n=0}^{\infty} \left(\frac{1}{2^n} + \frac{1}{2^n}\right)^2 = \sum_{n=0}^{\infty} \left(\frac{1}{2^{2n}}\right)^2 = \sum_{n=0}^{\infty} \frac{1}{16^n}$$
and the series on the right-hand side of the inequality is a convergent geometric series, the series converges by the Comparison Test. Since all of the terms are positive, the series converges absolutely.

(c) Since
$$\sum_{n=1}^{\infty} \left|\frac{\sin(1/n)}{n^3}\right| < \sum_{n=1}^{\infty} \frac{1}{n^3},$$
and the series on the right-hand side of the inequality is a convergent p-series, the series converges absolutely by the Comparison Test.

(d) Use the Ratio Test. Since
$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{2^{n+1}}{(n+1)!(\ln(n+1))^3} \cdot \frac{n!(\ln n)^3}{2^n} = \frac{2}{n} \cdot \left(\frac{\ln n}{\ln(n+1)}\right)^3,$$
we can determine that $\lim_{n\to\infty} |a_{n+1}/a_n| = 0 < 1$. Thus, the series converges absolutely.

(e) Since the terms, $a_n = (-1)^n/(n \ln n)$ are decreasing and $\lim_{n\to\infty} a_n = 0$, the series converges by the Alternating Series Test.

To determine the convergence of
$$\sum_{n=2}^{\infty} \frac{1}{n \ln n}$$
we use the Integral Test,
$$\int_2^{\infty} \frac{1}{x \ln x}\, dx = \int_{\ln 2}^{\infty} \frac{1}{u}\, du = \ln u \Big|_{\ln 2}^{\infty} = \infty.$$
Since the integral diverges, the series diverges. Thus, the series is conditionally convergent.

3. (20 points) Let f be a function having derivatives of all orders for all real numbers. The third-degree Taylor polynomial for f about $x = -2$ is given by

$$T_3(x) = 2 - \frac{3}{8}(x+2)^2 - \frac{1}{12}(x+2)^3.$$

(a) Find $f(-2)$, $f'(-2)$, and $f''(-2)$.

(b) Determine whether f has a local minimum, a local maximum, or neither at $x = -2$. Justify your answer.

(c) Use $T_3(x)$ to find an approximation for $f(0)$.

(d) The fourth derivative of f satisfies the inequality

$$\left|f^{(4)}(x)\right| \leq \frac{1}{4}$$

for all x in the closed interval $[-2, 0]$. Find an error bound on the approximation for $f(0)$ that you found in part (c).

Solution.

(a) $f(-2) = 2$, $f'(-2) = 0$, and $f''(-2) = -3/4$ (since $f''(-2)/2! = -3/8$).

(b) Since $f'(-2) = 0$, and $f''(-2) = -3/4$, f has a local a local maximum at $x = -2$ by the Second Derivative Test.

(c) $T_3(0) = -1/6$

(d) $|f(0) - T_3(0)| \leq \frac{M}{4!}|0+2|^4 = \frac{(1/4)}{24} \cdot 2^4 = \frac{1}{6}$

4. (6 points) Suppose we know that $\sum_{n=1}^{\infty} a_n$ converges to 0.8. We are given *no* other information about the infinite series. For each of the following statements circle

- *True* if the statement *must* be true,
- *False* if the statement *must* be false, and
- *Inconclusive* if the statement could be either true or false.

You do not need to justify your answers.

(a) $\lim_{n\to\infty} a_n = 0.8$ *True* *False* *Inconclusive*

(b) $\lim_{n\to\infty} a_n = 0$ *True* *False* *Inconclusive*

(c) $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} > 1$ *True* *False* *Inconclusive*

(d) $a_{n+1} < a_n$ for all n *True* *False* *Inconclusive*

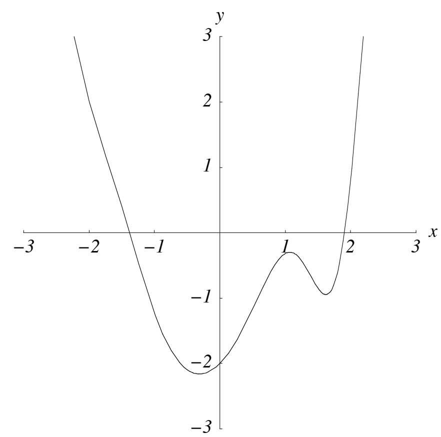
(e) $\lim_{n\to\infty} \frac{1}{|a_n|} = \infty$ *True* *False* *Inconclusive*

(f) $\lim_{n\to\infty} S_n = 0.8$, *True* *False* *Inconclusive*
where $S_n = a_1 + a_2 + \cdots + a_n$.

Solution. (a) *False*, (b) *True*, (c) *False*, (d) *Inconclusive*, (e) *True*, (f) *True*.

5. (6 points) The graph of $y = f(x)$ is given below. Assume that f is infinitely differentiable everywhere. The Taylor series for $f(x)$ about $x = 0$ is given by

$$f(x) = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots .$$



(a) Determine whether c_0 is positive, negative, or zero. Explain your reasoning.

(b) Determine whether c_1 is positive, negative, or zero. Explain your reasoning.

(c) Determine whether c_2 is positive, negative, or zero. Explain your reasoning.

Solution.

(a) Negative, $c_0 = f(0) \approx -2 < 0$

(b) Positive, $c_1 = f'(0) \approx 1 > 0$

(c) Positive, $c_2 = f''(0)/2! > 0$ since the function is concave up at $x = 0$.

6. (4 points) Consider the following power series.

(a) $x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \cdots$

(b) $1 - \dfrac{x^4}{2!} + \dfrac{x^8}{4!} - \cdots$

(c) $\dfrac{x^2}{2!} - \dfrac{x^4}{4!} + \dfrac{x^6}{6!} - \cdots$

(d) $x - x^3 + x^5 - \cdots$

(e) $x^2 - \dfrac{x^4}{2!} + \dfrac{x^6}{4!} - \cdots$

Fill in the letter of the series that corresponds to the given function. *You do not need to justify your answer.*

______ $1 - \cos x$

______ $x^2 \cos x$

______ $\sin(x^2)$

______ $\dfrac{x}{1+x^2}$

Solution. $1 - \cos x$ is (c); $x^2 \cos x$ is (e); $\sin(x^2)$ is (a); $\dfrac{x}{1+x^2}$ is (d).

7. (10 points) Find the interval of convergence for the power series

$$\sum_{n=1}^{\infty} \frac{(x+5)^n}{n^2 3^n}.$$

If the interval of convergence is finite, make sure that you determine the convergence at each endpoint and justify your conclusions.

Solution. Using the Ratio Test,

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(x+5)^{n+1}}{(n+1)^2 3^{n+1}} \cdot \frac{n^2 3^n}{(x+5)^n}\right| = \frac{n^2}{3(n+1)^2}|x+5|,$$

we can determine that $\lim_{n\to\infty} |a_{n+1}/a_n| = |x+5|/3$. This limit is less than one on the interval $|x+5| < 3$ or $-8 < x < -2$, and the series converges on this interval.

We must consider the endpoints separately. At $x = -2$, the series

$$\sum_{n=1}^{\infty} \frac{3^n}{n^2 3^n} = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

is a convergent p-series. At $x = -8$, the series

$$\sum_{n=1}^{\infty} \frac{(-3)^n}{n^2 3^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$$

converges by the Alternating Series Test.

Thus, the power series converges on the interval $-8 \le x \le -2$.

8. (20 points)

   (a) Write down a power series expansion for e^{-x^3}.

   (b) Write down a power series expansion for $\int e^{-x^3}\,dx$ and determine its radius of convergence.

   (c) Use your answer in part (b) to find a series for $\displaystyle\int_0^{1/2} e^{-x^3}\,dx$.

   (d) If you approximate the definite integral $\int_0^{1/2} e^{-x^3}\,dx$ by taking the partial sum consisting of the first four nonzero terms of the series that you obtained in part (c), what is the maximum error for your approximation.

Solution.

(a) Since

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!},$$

the power series expansion for e^{-x^3} is

$$e^{-x^3} = 1 - x^3 + \frac{x^6}{2!} - \frac{x^9}{3!} + \frac{x^{12}}{4!} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{3n}}{n!}.$$

(b) To find a power series expansion for $\int e^{-x^3}\,dx$, we integrate the power series obtained in (a),

$$\int e^{-x^3}\,dx = c_0 + x - \frac{x^4}{4} + \frac{x^7}{7\cdot 2!} - \frac{x^{10}}{10\cdot 3!} + \frac{x^{13}}{13\cdot 4!} - \cdots = c_0 + \sum_{n=0}^{\infty} \frac{(-1)^n x^{3n+1}}{(3n+1)\cdot n!}$$

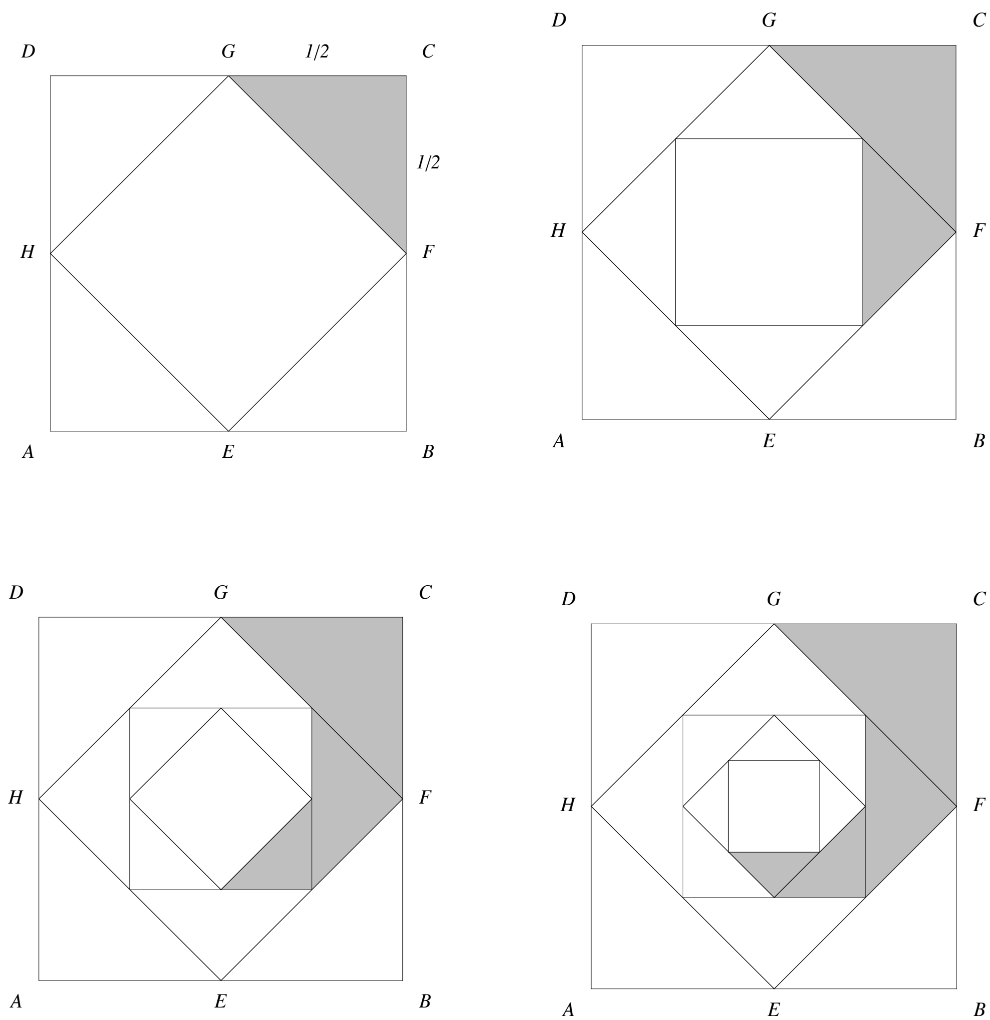
where c_0 is an arbitrary constant. The radius of convergence does not change—$R = \infty$.

(c) $$\int_0^{1/2} e^{-x^3}\,dx = \frac{1}{2} - \frac{1}{2^4\cdot 4} + \frac{1}{2^7\cdot 7\cdot 2!} - \frac{1}{2^{10}\cdot 10\cdot 3!} + \frac{1}{2^{13}\cdot 13\cdot 4!} - \cdots$$

(d) Since this is an alternating series, the maximum error is bounded by the fifth term of the series,

$$\frac{1}{2^{13}\cdot 13\cdot 4!}$$

9. (8 points) Assume that the square $ABCD$ below has sides of *length one* and that E, F, G, and H are the midpoints of the sides.



(a) If the indicated pattern is continued indefinitely, write an infinite series that will give the area of the shaded region?

(b) What is the area of the shaded region if the indicated pattern is continued indefinitely? That is, what is the sum of the series that you found in part (a)?

Solution.

(a) $\dfrac{1}{8} + \dfrac{1}{16} + \dfrac{1}{32} + \dfrac{1}{64} + \dfrac{1}{128} + \cdots = \dfrac{1}{2^3} + \dfrac{1}{2^4} + \dfrac{1}{2^5} + \dfrac{1}{2^6} + \dfrac{1}{2^7} + \cdots$

(b) The series in (a) is geometric. Thus,

$$\frac{1}{2^3} + \frac{1}{2^4} + \frac{1}{2^5} + \frac{1}{2^6} + \frac{1}{2^7} + \cdots = \frac{1}{2^3} \cdot \frac{1}{1 - (1/2)} = \frac{1}{4}.$$