Review Problems for the Series Exam

1 *Some Expectations:*

- **Conceptual understanding of series:** What does it mean for a series to converge? What are criteria for convergence? What is a power series and what are the possibilites for the interval of convergence of a power series?
- **Taylor polynomials, Taylor series:** Understand the relationship between the graph of a function near a point and the first few terms of the Taylor series at that point. Be able to compute the Taylor series for f at $x = a$ directly by taking derivatives, evaluating at a and establishing a pattern if possible. Be able to obtain a Taylor series from a familiar Taylor series via substitution, integration, and differentiation. Have some sense as to when to compute the series from scratch and when and how to manipulate a known series.
- **Convergence tests:** Know the tests and how and when to apply them
- **Determining radius of convergence of power series and interval of convergence**
- **Geometric series:** Be able to set up and solve a problem involving a geometric series and/or geometric sums.
- **Approximations** Be able to find reasonable upper bounds for the remainder using AST error estimate or the Taylor remainder. (We will only use the latter in the simplest cases.)

2 Review Problems:

2.1 Conceptual Understanding of Convergence

1. Suppose you know that the series $\sum_{n=1}^{\infty} a_n$ converges absolutely. For each of the following series, determine whether the series converges, diverges, or there is insufficient information to determine whether or not the series converges. Explain your reasoning carefully.

   (a) $\sum_{n=1}^{\infty}(a_n)^2$

   (b) $\sum_{n=1}^{\infty}(a_n)^3$

   (c) $\sum_{n=1}^{\infty} \frac{1}{(a_n)^2}$

   (d) $\sum_{n=1}^{\infty} \frac{2^n a_n}{3^n}$

2. Suppose you know that the series $\sum_{n=1}^{\infty}(-1)^n a_n$ converges and that all the a_n are positive, $n = 1, 2, 3, \ldots$. For each of the following series, determine whether the series converges, diverges, or there is insufficient information to determine whether or not the series converges. Explain your reasoning carefully.

   (a) $\sum_{n=1}^{\infty} a_n$
   (b) $\sum_{nk=1}^{\infty} \frac{a_n}{n}$
   (c) $\sum_{n=1}^{\infty}(-1)^n \frac{10a_n}{3}$
   (d) $\sum_{n=1}^{\infty}(-1)^n na_n$

3. What does it mean to write $\sum_{n=1}^{\infty} b_n = 11$?

4. Suppose that $\displaystyle\sum_{n=0}^{\infty} a_n$ converges, and that $a_0, a_1, \ldots a_n, \ldots$ are all positive. Which of the following series *must* converge? Which of them *must* diverge? If it cannot be determined whether the series converges or diverges given the information, indicate this.

   (i) $\displaystyle\sum_{n=0}^{\infty} \frac{a_n}{2^n}$ (ii) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{a_n}$ (iii) $\displaystyle\sum_{n=0}^{\infty} (\sin n)\cdot a_n$ (iv) $\displaystyle\sum_{n=0}^{\infty} \frac{5^n a_n}{4^n}$

2.2 Taylor Polynomials, Taylor Series

5. (a) Suppose $f(x)$ is an even function: $f(-x) = f(x)$ for all x. Show that the Taylor series expansion of f at $x = 0$ has only even powers of x.

   (b) Suppose $f(x)$ is an odd function: $f(-x) = -f(x)$ for all x. Show that the Taylor series expansion of f at $x = 0$ has only odd powers of x.

   Hints: The derivative of an odd function is even, and the derivative of an even function is odd. If $g(x)$ is an odd function, what must $g(0)$ be?

6. Find the 6th degree polynomial that best approximates $\frac{\sin(x^2)}{x}$ for x near 0.

7. (a) Find the Taylor series centered about $x = 0$ for the function $5x \cos x^3$. Either include a general term for your series or use summation notation – whichever you prefer.

   (b) Find the 7th degree Taylor polynomisal generated by $\sin x$ about $x = \pi/4$.

   (c) Write the first four non-zero terms of the series expansion of $\int_0^{0.2} 5x \cos x^3 dx$.

   (d) How many (non-zero) terms of the series are needed in order to compute $\int_0^{0.2} 5x \cos x^3 dx$ with an error of less than $\frac{1}{10^6}$? Explain your reasoning.

8. (a) Construct the Taylor series representation of the function $f(x) = \frac{1}{1-x}$ at $x = 10$. Include a general term or write your answer in summation notation.

   (b) What is the radius of convergence of this power series?

   (c) If you evaluate the power series from part (a) at $x = 17$ will the series converge to a sum of $\frac{1}{1-17} = \frac{1}{-16}$? Explain very briefly.

   (d) If you evaluate the power series from part (a) at $x = \frac{1}{2}$ will the series converge to a sum of $\frac{1}{1-\frac{1}{2}} = 2$? Explain very briefly.

9. $f(x) = \sqrt{3} + 12(x-5)^3 + 17(x-5)^6$

   Find the following:

   (a) $f(5)$ (b) $f''(5)$ (c) $f'''(5)$ (d) $f^{(6)}(5)$

10. Compute the 6th degree Taylor polynomial generated by $\cos x$ about $x = -\pi/2$.

11. The functions $\cosh x$ and $\sinh x$ are defined as follows:

$$\cosh x = \frac{e^x + e^{-x}}{2} \qquad \sinh x = \frac{e^x - e^{-x}}{2}$$

   (a) Graph $\cosh x$ and $\sinh x$ with or without a graphing calculator. (At this point you'll wonder why the names for these functions are so close to those of the trigonometric functions.)

   (b) What is the derivative of $\sinh x$? of $\cosh x$?

   (c) Find the Taylor series for $\sinh x$ about $x = 0$.

   (d) Use your answer to (c) to find the Taylor series for $\cosh x$ about $x = 0$.

12. (a) Find the Taylor series about $x = 0$ for $\ln(\frac{1+x}{1-x})$ by subtracting the Taylor series for $\ln(1-x)$ at $x = 0$ from that of $\ln(1+x)$.

   (b) Show that when $x = 1/3$, $\frac{1+x}{1-x} = 2$.

   (c) Use the first four nonzero terms of the series in part (a) to approximate $\ln 2$. compare your answer with the approximation given by the first four terms of the series for $\ln(1+x)$ evaluated at $x = 1$ and the value of $\ln 2$ given by a computer or calculator.

2.3 Convergence Tests

13. For each of the following infinite series, determine if it converges or diverges. Mathematically justify your answer.

(a) $\sum_{n=1}^{\infty} \left(\frac{e}{\pi}\right)^{(n-1)}$ (b) $\sum_{n=1}^{\infty} \frac{2n^2+1}{3n^2+n}$ (c) $\sum_{n=1}^{\infty} \frac{(-1)^n \pi}{\sqrt{3n}}$

(d) $\sum_{n=1}^{\infty} \frac{3^n}{4^n\, n!}$ (e) $\sum_{n=1}^{\infty} \frac{\sqrt{3}}{n(n-4)}$ (f) $\sum_{n=1}^{\infty} \frac{\cos n}{1.1^n}$

14. For any real number p, the alternating p–series is the infinite series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^p} = 1 - \frac{1}{2^p} + \frac{1}{3^p} - \frac{1}{4^p} + \cdots$$

    (a) For what values of p does the alternating p–series diverge?
    (b) For what values of p does the alternating p–series is converge? For which of these values of p does it converge absolutely?

2.4 Radius and interval of convergence

15. Let $f(x) = \sum_{n=1}^{\infty} \frac{x^n}{n \ln(n+1)}$.
    (a) What is the radius of convergence of this power series?
    (b) What is the interval of convergence of this power series?

16. What is the interval of convergence of $\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$

17. What is the interval of convergence of $\sum_{n=1}^{\infty} \frac{(2x-5)^n}{10^{n+1}}$

18. Give examples of power series such that the radius of convergence is

    (a) $R = 0$ (b) $R = \infty$ (c) $R = 5$

19. Give an example of a series whose interval of convergence is $(0, 12)$.

2.5 Geometric Series

20. One farmer decided to produce honey at his farm. He bought a beehive for 364 bees and only one bee. At the end of the first day the bee felt so lonely, that the next day it found 3 new roommates. But the newcomers also felt lonely and the next day each of them found 3 new roommates. At the end of which day the beehive was full, if every newcomer found 3 new roommates the next day after moving in beehive itself?

21. Consider the nested squares s_0, s_1, s_2, ... created as follows. s_0 is a square whose sides are length 2. s_0 is the square whose vertices are the midpoints of the side of s_0. (Join these vertices to create square s_1 living inside square s_0.) s_2 is the square whose vertices are the midpoints of the side of s_1. (Join these vertices to create square s_2 living inside square s_1.) And so on. s_{n+1} is obtained by joining the midpoints of the edges of s_n.

    (a) Calculate the areas of squares s_0, s_1, s_2, and s_3.

    (b) Find the sum of the areas of all the squares. (There are infinitely many squares.)

2.6 Approximations

22. How many non-zero terms of the Taylor series for e^x about $x = 0$ are needed to approximate $\frac{1}{e}$ with error less than 10^{-3}?

23. If the approximation $\cos x \approx 1 - x^2/2 + x^4/24$ is used for values of x such that $|x| < 0.5$ what is a reasonable upper bound for the error that could be incurred?

24. The equation $\pi/4 = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots$ comes from evaluating the series for $\arctan x$ centered about $x = 0$ at $x = 1$. How many terms are needed in order to approximate $\pi/4$ with error less than 0.01?

25. (a) Approximate $f(x) = \sqrt{1 + x}$ by the first three non-zero terms of its Taylor series about $x = 0$.

    (b) Find a reasonable bound for the error in approximating in using the approximation in (a) for $f(.2)$.

    (c) Find a reasonable bound for the error in approximating in using the approximation in (a) for $f(-.3)$.