Name:______________________________

Math 1b Final Exam
Tuesday, May 23, 2006

Please circle your section:

Rina Anno 10–11 MWF	Li-Sheng Tseng 10–11 MWF	Robert Strain 11–12 MWF
Thomas Judson 11–12 MWF	Robin Gottlieb 10–11:30 TTh	Robin Gottlieb 11:30–1 TTh

Problem Number	Possible Points	Score
1	9	
2	4	
3	7	
4	8	
5	8	
6	15	
7	8	
8	4	
9	9	
10	8	
11	6	
12	6	
13	8	
Total	100	

Directions—Please Read Carefully! You have three hours to take this exam. Make sure to use correct mathematical notation. To receive full credit on a problem, you will need to justify your answers carefully—unsubstantiated answers will receive little or no credit. Please be sure to write neatly—illegible answers will receive little or no credit. If more space is needed, use the back of the previous page to continue your work. Be sure to make a note of this on the problem page so that the grader knows where to find your answers. *Calculators are not allowed.* **Good Luck!!!**

1. (9 points) Evaluate the following integrals

(a) $\displaystyle\int \frac{5x+7}{(x+1)(x+2)}\,dx$

(b) $\displaystyle\int_0^1 x\arctan x\,dx$

(c) $\displaystyle\int_2^\infty \frac{1}{x(\ln x)^2}\,dx$

2. (4 points) Suppose that you wish to model a population with a differential equation of the form $dP/dt = f(P)$, where $P(t)$ is the population at time t. Experiments have been performed on the population that give the following information:

- The population at $P = 0$ remains constant.
- A population of $0 < P < 20$ will decrease.
- A population of $P = 20$ does not change.
- A population of $20 < P < 100$ increases.
- A population of $P > 100$ will decrease.

Which of the following differential equations best models this population? ***Circle the correct answer.***

(a) $\frac{dP}{dt} = (P - 20)(P - 100)$

(b) $\frac{dP}{dt} = P(20 - P)(P - 100)$

(c) $\frac{dP}{dt} = P(P - 20)(100 - P)$

(d) $\frac{dP}{dt} = P(20 - P)(100 - P)$

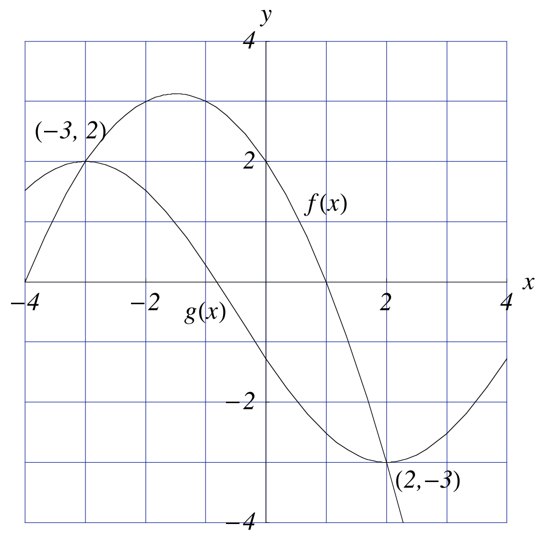
(e) $\frac{dP}{dt} = (20 - P)(P - 100)$

3. (7 points) A bag of sand originally weighing 144 lb is lifted at a constant rate. As it rises, sand leaks out at a constant rate. The sand is half gone by the time the bag has been lifted 18 ft.

   (a) How many pounds of sand leak out of the bag ***per foot*** as the bag is lifted?

   (b) How much work was done in lifting the bag 18 ft? To receive full credit for your work, indicate clearly what your variable is and what you are considering to be zero.

4. (8 points) Let R be the region bounded by the curves $y = f(x)$ and $y = g(x)$ shown in the graph below.

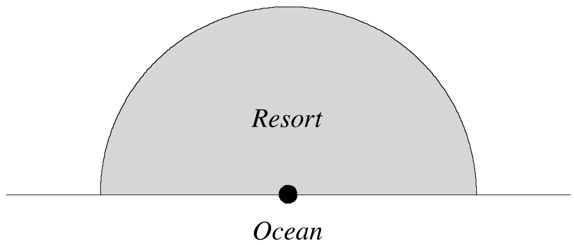


(a) Write a definite integral that will give the area of the region R.

(b) Write a definite integral that will give the volume of the solid generated when the region R is revolved about the horizontal line $y = 4$.

(c) If the base of a solid V is the region R and the cross-sections of the solid perpendicular to the x-axis are squares, write a definite integral that will give the the volume of V.

5. (8 points) A resort town is laid out along the seashore in the shape of a semicircle of radius 3 miles with the diameter of the semicircle bordering the ocean. People want to live close to the center of the town (indicated by the solid black disk). The density of the population (individuals per square mile) at a distance of x miles from the center of town is given by $\rho(x) = 600 - 200x$.



(a) Write a Riemann sum that approximates the total population of the resort.

(b) Use your answer from part (a) to write a definite integral that represents the total population of the town and evaluate the integral.

6. (15 points) In an isolated region of the Canadian Northwest Territories, a population of arctic wolves and a population of silver foxes compete for resources. The two species have a common, limited food supply, which consists mainly of mice. If

$$
\begin{aligned}
x(t) &= \text{the population of arctic wolves (in thousands)} \\
y(t) &= \text{the population of silver foxes (in thousands),}
\end{aligned}
$$

the interaction of the two species can be modeled by the following system of differential equations,

$$
\begin{aligned}
\frac{dx}{dt} &= x - x^2 - xy \\
\frac{dy}{dt} &= \frac{3}{4}y - y^2 - \frac{1}{2}xy.
\end{aligned}
$$

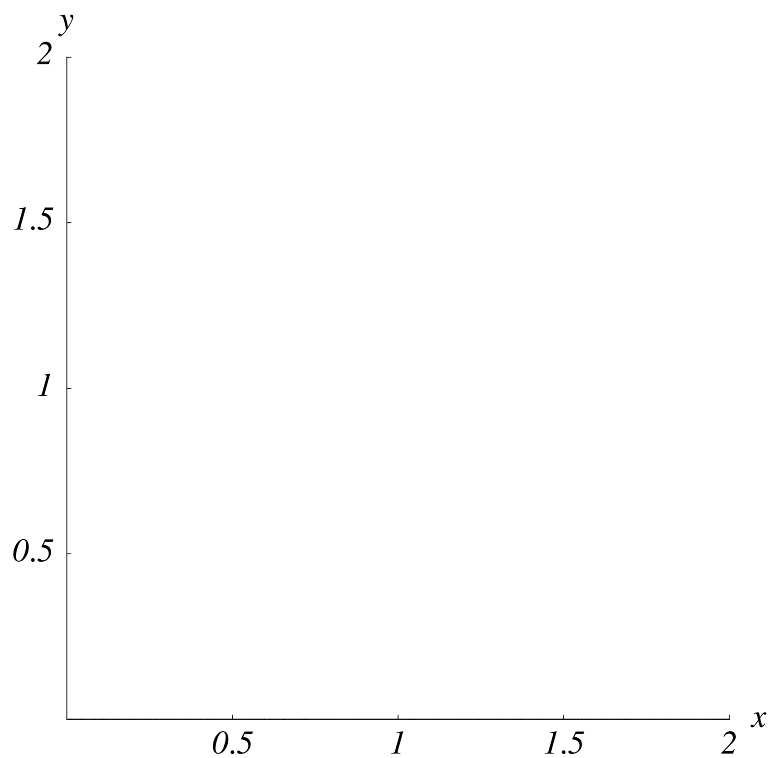
(a) Find the nullclines of the system for $x \geq 0$ and $y \geq 0$. (Axes are provided on the next page.)

(b) Find all of the equilibrium points for $x \geq 0$ and $y \geq 0$.

$$\frac{dx}{dt} = x - x^2 - xy, \qquad \frac{dy}{dt} = \frac{3}{4}y - y^2 - \frac{1}{2}xy.$$

(c) What happens to the arctic wolf population in the absence of silver foxes? What happens to the silver fox population in the absence of arctic wolves?

(d) Sketch and label the nullclines for $x \geq 0$ and $y \geq 0$. Be sure to indicate the direction of the solution on the nullclines and in the regions bounded by the nullclines.



(e) Sketch the solution trajectory with the initial conditions $x(0) = 0.5$ and $y(0) = 1.5$, indicating the direction of your solution curve.

7. (8 points) A dosage d of a drug is given daily at $t = 0, 1, 2, 3, \ldots$ days. The drug decays exponentially at a rate r in the blood stream. Thus, the amount in the bloodstream after $n+1$ doses is $d + de^{-r} + de^{-2r} + \cdots + de^{-nr}$

   (a) Find the level of the drug after an "infinite" number of doses. That is, find $d + de^{-r} + de^{-2r} + \cdots + de^{-nr} + \cdots$

   (b) If $r = 0.1$, what dosage is needed to maintain a drug level of 2?

8. (4 points) The following polynomials are second-degree Taylor polynomials for functions whose graphs are given below. Match each Taylor polynomial with the appropriate graph.

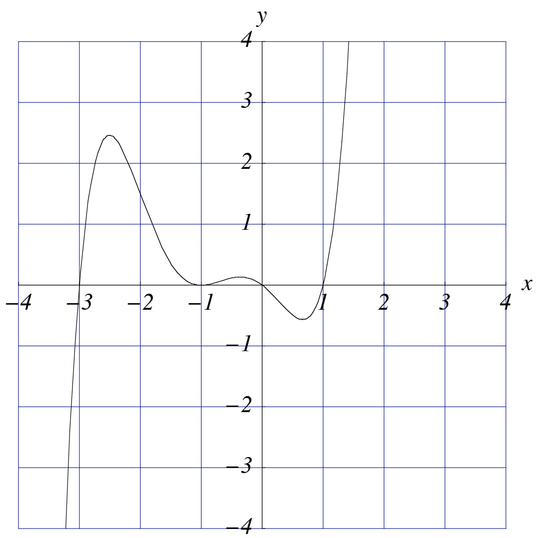
(a) $T_2(x) = -2(x-1) - 2(x-1)^2$

(b) $T_2(x) = 2 + 2(x-1) - \frac{5}{2}(x-1)^2$

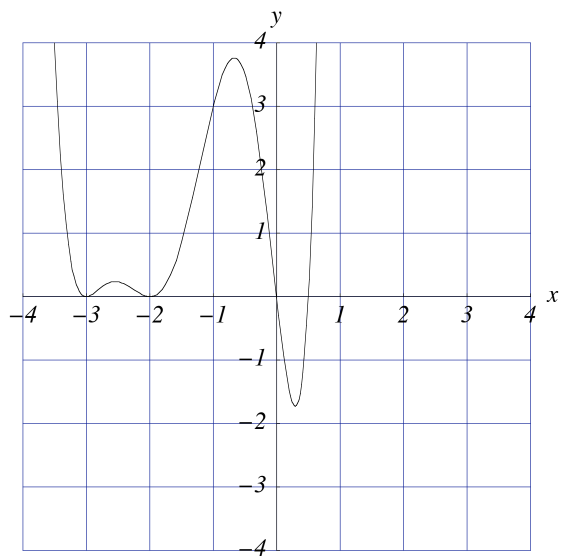
(c) $T_2(x) = 4(x-1) + 9(x-1)^2$

(d) $T_2(x) = 3 + \frac{8}{3}(x+1) - \frac{13}{6}(x+1)^2$

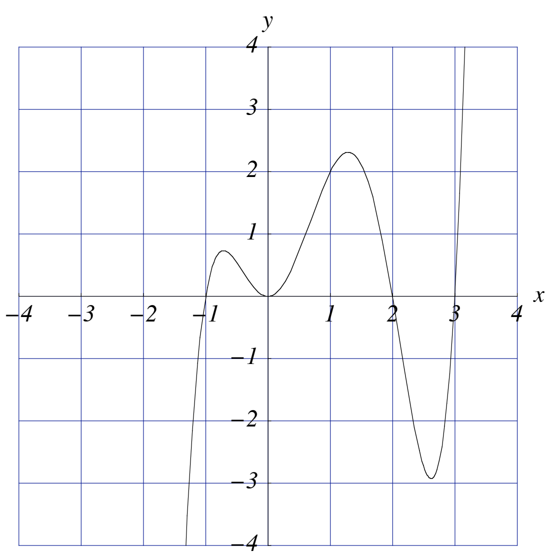
(i) ______	(ii) ______	(iii) ______	(iv) ______



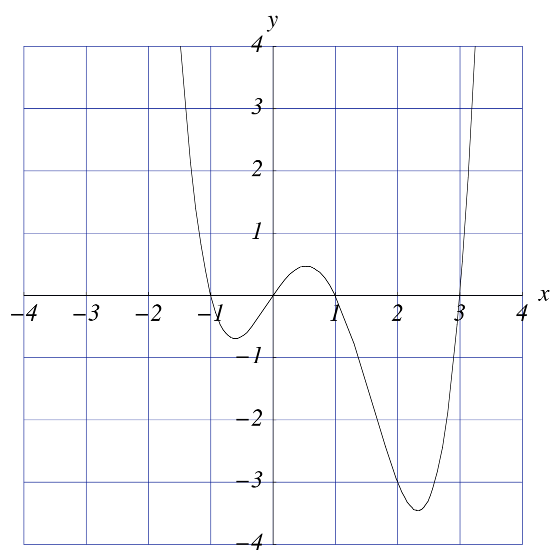
(i)



(ii)



(iii)



(iv)

9. (9 points) Find a power series representation at $x = 0$ for each of the following functions.

(a) $\dfrac{x}{1-x}$

(b) $x^2 \cos x$

(c) $\displaystyle\int_0^x t^2 \sin t^2 \, dt$

10. (8 points) Find the interval of convergence for the power series

$$\sum_{n=1}^{\infty} \frac{(x+5)^n}{\sqrt{n}3^n}.$$

If the interval of convergence is finite, make sure that you determine the convergence at each endpoint and justify your conclusions for the convergence or divergence at the endpoints.

11. (6 points) Match each slope field with one of the following differential equations.

(a) $\dfrac{dy}{dt} = ty^2$

(b) $\dfrac{dy}{dt} = y - t$

(c) $\dfrac{dy}{dt} = ty$

(d) $\dfrac{dy}{dt} = 1 - y$

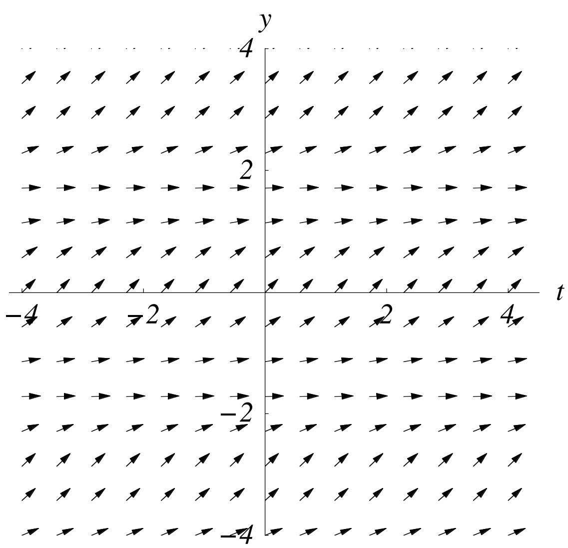
(e) $\dfrac{dy}{dt} = \cos^2 y$

(f) $\dfrac{dy}{dt} = y(y^2 - 1)$

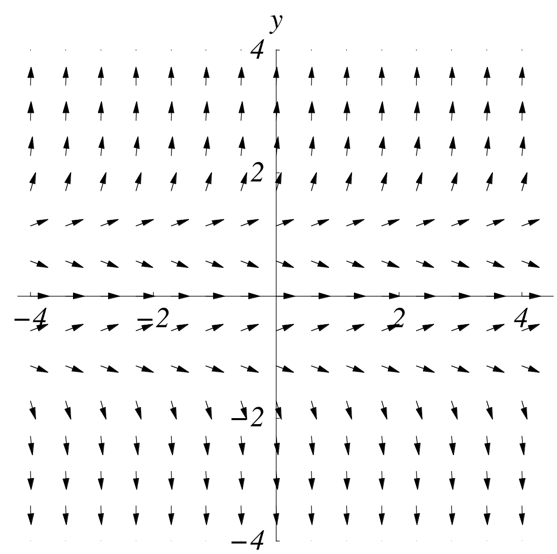
(g) $\dfrac{dy}{dt} = \cos^2 t$

(h) $\dfrac{dy}{dt} = y(y - 1)$

(i) ______	(ii) ______	(iii) ______	(iv) ______



(i)



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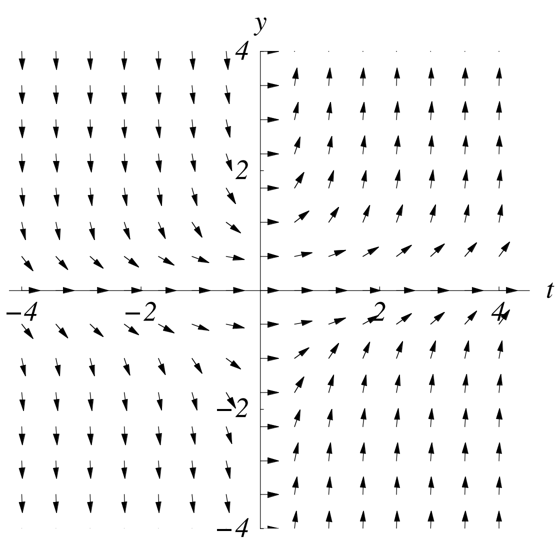

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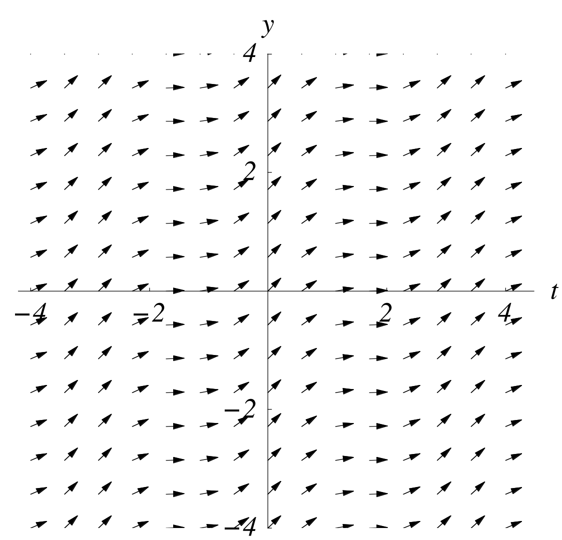


(iv)

12. (6 points) Match each of the following graphs of y versus t to the differential equation for which it could be a solution.

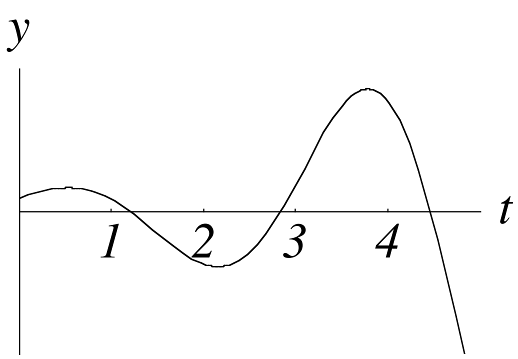
(a) $y'' + 5y' + 4y = 0$

(b) $y'' - 5y' + 4y = 0$

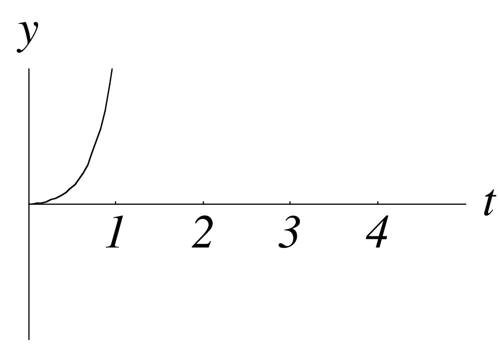
(c) $y'' - y' + 4y = 0$

(d) $y'' + y' + 4y = 0$

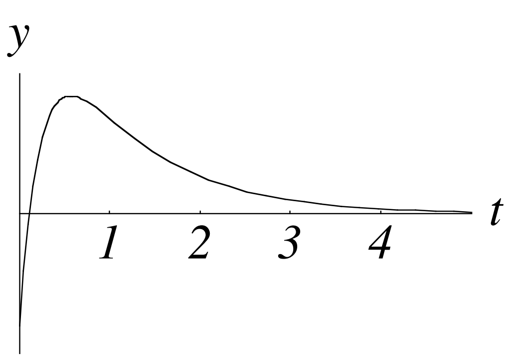
(i) ______	(ii) ______	(iii) ______	(iv) ______



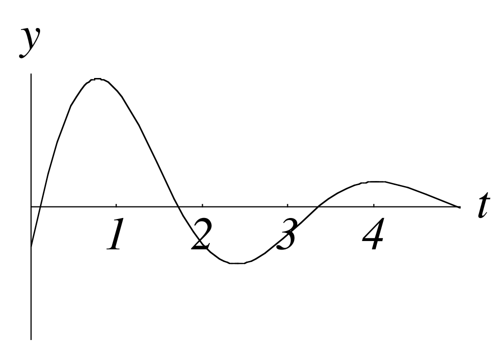
(i)



(ii)



(iii)



(iv)

13. (8 points) A new 15-gallon juice dispenser in the cafeteria is filled with a fruit drink that is 80% orange juice and 20% pineapple juice. Every hour, 10 gallons of juice is consumed. But due to an error, the dispenser is continuously replenished every hour with an orange-pineapple mixture that is 40% orange and 60% pineapple. Assume that the dispensed juice is always well-mixed.

(a) Write down a differential equation for $P(t)$, the amount of pineapple juice in the dispenser at time t. Be sure to include your initial condition.

(b) By solving the differential equation, find the amount of pineapple juice in the container after 2 hours.