Harvard University
Math 1b. Second Semester Calculus — Name ____________________
First Exam — October 28, 1998

First Exam

- Do not open this exam booklet until you are directed to do so.
- You have 120 minutes to earn 80 points.
- Write your solutions in the space provided. If you need more space, write on the back of the sheet containing the problem. Do not put part of the answer to one problem on the back of the sheet for another problem.
- Do not spend too much time on any problem. Read them all through first and attack them in the order that allows you to make the most progress.
- Show your work, as partial credit will be given. You will be graded not only on the correctness of your answer, but also on the clarity with which you express it. Be neat.
- Good luck!

Problem	Points	Grade
1	15	
2	10	
3	10	
4	10	
5	10	
6	15	
7	10	
Total	80	

Please circle your section:

MWF 10:00 Brian Conrad	MWF 10:00 Andy Engelward	MWF 10:00 Eric Towne	MWF 11:00 Noam Elkies	MWF 11:00 Robert Pollack	MWF 12:00 Andy Engelward
	TTh 10:00 Tomas Klenke	TTh 10:00 Joel Rosenberg	TTh 10:00 Eric Towne	TTh 11:30 Heather Russell	

1. (15 pts)

1a. Determine whether each of the following series converges or diverges. Justify your answers.

(i) $\displaystyle\sum_{k=1}^{\infty}\left(1-\frac{1}{k^2}\right)$

(ii) $\displaystyle\sum_{n=0}^{\infty}\frac{n^2+1}{n!}$

(iii) $\displaystyle\sum_{k=1}^{\infty}\frac{\cos^2 k}{k^2+1}$

1b. For what real values of a do the following series converge? Explain your answers.

(i) $\sum_{k=1}^{\infty} a$

(ii) $\sum_{k=1}^{\infty} (-1)^k k^a$

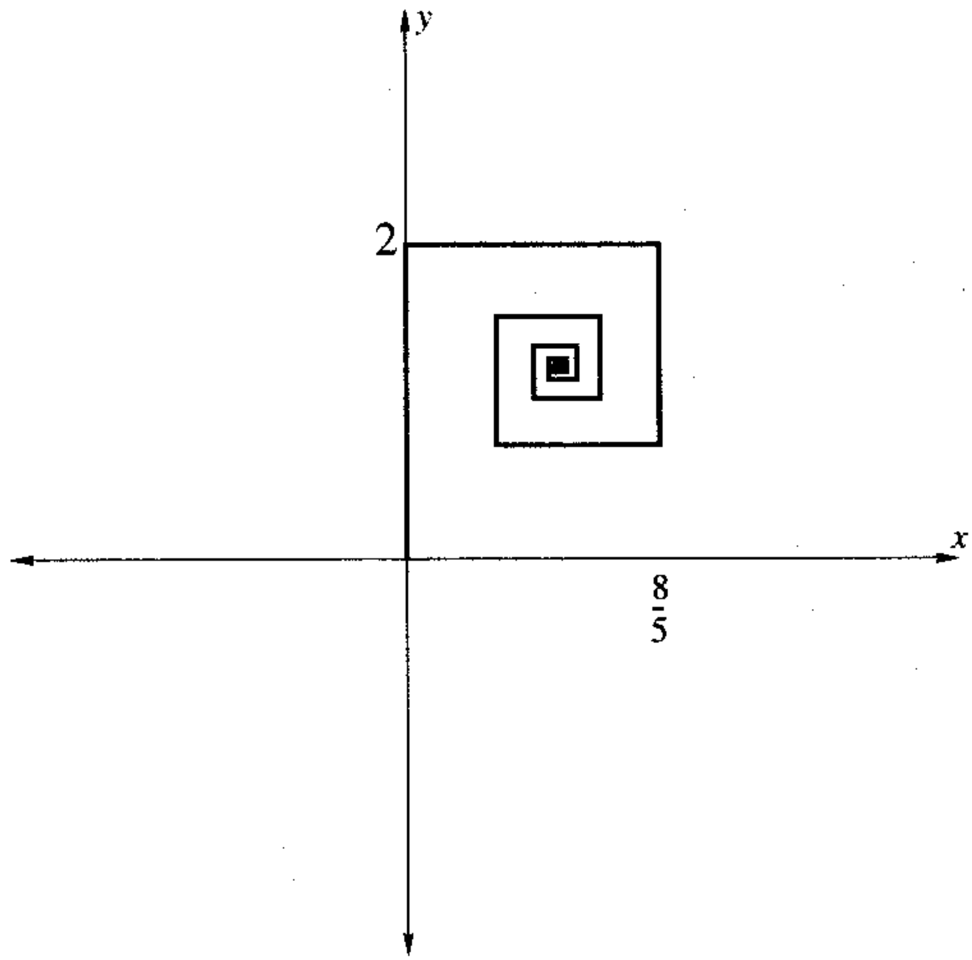
2. (10 pts) For each power series below, give the radius and interval of convergence. Be sure to check convergence at the endpoints.

a. $\displaystyle\sum_{k=0}^{\infty} \frac{k!(x-3)^k}{2^k}$

b. $\displaystyle\sum_{k=1}^{\infty} \frac{(x-4)^k}{k \cdot 3^k}$

3. (10 pts) I want to draw a spiral, but since I am not that good at drawing curves, I've decided to draw a "spiral with corners" instead, as follows:

I start at the origin $(0,0)$ of the xy-plane, and draw a segment 2 units long, vertically upwards. Then I turn right by $90°$, and draw a segment $\frac{4}{5}$ as long as the first segment. Then I make another $90°$ right turn, and draw a segment $\frac{4}{5}$ as long as the second segment. I continue this way, and get my spiral:



(Question continues on next page)

a. What is the total distance my pen will eventually travel?

b. Find the coordinates (x_0, y_0) of the point P in the xy-plane at which I will eventually end up. (Hint: consider the x-coordinate after each segment is drawn, and then consider the y-coordinate after each segment is drawn.)

4. (10 pts)

a. Find the second degree Taylor polynomial for $f(x) = \sqrt{x}$ centered at $x = 4$.

b. Use your answer to **a** to give an approximation to $\sqrt{3.9}$. (You need not simplify your answer.)

c. If you wanted to approximate $\sqrt[3]{31}$ by using a Taylor series for $\sqrt[3]{x}$, where would you center your Taylor polynomial? Explain your reasoning briefly. Do not attempt to approximate $\sqrt[3]{31}$; just say how you would do it.

5. (10 pts) Show that if we try to compute $\sqrt[3]{1.06}$ by approximating the function

$$g(x) = \sqrt[3]{1+x}$$

by the polynomial $1 + \frac{1}{3}x$, then the error will be less than .0005.

6. (15 pts) Let $f(x)$ be given by the power series

$$f(x) = \sum_{k=1}^{\infty} \frac{(x-2)^k}{\sqrt{k}}.$$

a. Determine the radius of convergence and the interval of convergence of this series. Be sure to check convergence at the endpoints.

b. Write out the first four nonzero terms of the power series for the derivative $\frac{df}{dx}$.

What are the radius and interval of convergence of the power series for $\frac{df}{dx}$?

c. Let $g(x)$ be an antiderivative for $f(x)$, i.e. $\dfrac{dg}{dx} = f(x)$, and such that $g(2) = 0$. What are the first four nonzero terms of the power series for $g(x)$?
What are the radius and interval of convergence of the series for $g(x)$?

7. (10 pts)

a. Give the first four nonzero terms of the Taylor series at $x = 0$ for the function

$$f(x) = \frac{e^{x^2} + e^{-x^2}}{2}.$$

b. Find the Maclaurin series for $x - \sin x$. (Write out the first four nonzero terms.)

c. Using your answer to **b**, compute

$$\lim_{x \to 0} \frac{x - \sin x}{x^3}.$$