Answers for May 21, 1999 Math 1B Final Exam

① (a) Diverges: use divergence test $\lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n}+10} = 1 \neq 0$

(b) Converges: use ratio test $\lim_{k\to\infty} \frac{(k+1)+2}{(k+1)!}\cdot\frac{k!}{k+2} = \lim_{k\to\infty} \frac{k+3}{(k+1)(k+2)}$

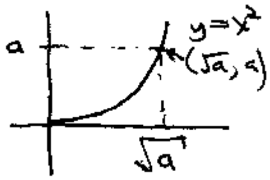
$= \lim_{k\to\infty} \frac{k+3}{k^2+3k+2}$ which by L'Hopital $= \lim_{k\to\infty} \frac{1}{2k+3} = 0 < 1$

(c) Converges: $= \sum_{n=1}^{\infty} \left(\frac{1}{e}\right)^n$ which is a geometric series with ratio $\frac{1}{e} < 1$

(d) $\sum_{n=1}^{\infty} \frac{\pi^2 n^3 - 15n}{n^5} = \sum_{n=1}^{\infty} \frac{\pi^2}{n^2} - \sum_{n=1}^{\infty} \frac{15}{n^4} = \pi^2\left(\sum_{n=1}^{\infty} \frac{1}{n^2}\right) - 15\left(\sum_{n=1}^{\infty} \frac{1}{n^4}\right)$

$= \pi^2\left(\frac{\pi^2}{6}\right) - 15\left(\frac{\pi^4}{90}\right)$

$= \frac{\pi^4}{6} - \frac{\pi^4}{6} = 0$

② (a) 

or $\sqrt{y} = x$

so Volume $= \int_0^a \pi (\sqrt{y})^2\, dy = \int_0^a \pi y\, dy$

$= \pi \frac{y^2}{2}\Big|_0^a = \frac{\pi a^2}{2}$

(b) Surface Area $= \int_0^a 2\pi f(y)\sqrt{1+(f'(y))^2}\, dy$

here $f(y) = y^{1/2}$ $\quad f'(y) = \frac{1}{2}y^{-1/2}$ $\quad (f'(y))^2 = \frac{1}{4y}$

so surface area $= \int_0^a 2\pi (y^{1/2})\sqrt{1+\frac{1}{4y}}\, dy = \int_0^a 2\pi\sqrt{y+\frac{1}{4}}\, dy$

sub $u = y + 1/4$ $\quad du = dy$ $\quad$ surface area $= \int_{y=0}^{y=a} 2\pi\sqrt{u}\, du$

$= 2\pi\left(\frac{2}{3}u^{3/2}\right)\Big|_{y=0}^{y=a} = \frac{4\pi}{3}\left(\sqrt{\left(y+\frac{1}{4}\right)^3}\right)\Big|_0^a$

$= \frac{4\pi}{3}\left(\left(a+\frac{1}{4}\right)^{3/2} - \left(\frac{1}{4}\right)^{3/2}\right)$

(3) (a) $y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \cdots$ $\quad a_0 = 1$ since $y(0)=1$

$y' = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + 5a_5 x^4 + \cdots$ $\quad a_1 = 2$ since $y'(0) = 2$

$y'' = 2a_2 + 6a_3 x + 12a_4 x^2 + 20a_5 x^3 + \cdots$

$-3y' = -3a_1 - 6a_2 x - 9a_3 x^2 - 12a_4 x^3 - 15a_5 x^4 - \cdots$

$+2y = 2a_0 + 2a_1 x + 2a_2 x^2 + 2a_3 x^3 + 2a_4 x^4 + 2a_5 x^5 + \cdots$

so $2a_2 - 3a_1 + 2a_0 = 0$ or $2a_2 - 3\cdot 2 + 2\cdot 1 = 0 \Rightarrow a_2 = 2$

$6a_3 - 6a_2 + 2a_1 = 0$ or $6a_3 - 12 + 4 = 0 \Rightarrow a_3 = \frac{4}{3}$

$12a_4 - 9a_3 + 2a_2 = 0$ or $12a_4 - 12 + 4 = 0 \Rightarrow a_4 = \frac{2}{3}$

so $y = 1 + 2x + 2x^2 + \frac{4x^3}{3} + \frac{2x^4}{3} + \cdots$

(b) this is $y = e^{2x}$

(4) (a) sub $u = \ln x$ $\quad du = \frac{1}{x}dx$ $\quad \int \frac{dx}{x}(\ln x)^{-1} = \int u^{-1}\,du$

$= \ln|u| + C = \ln|\ln x| + C$

(b) $\frac{d}{dx}\left(\ln|\ln x| + C\right) = \frac{1}{\ln x}\cdot\frac{d}{dx}(\ln x) = \frac{1}{(\ln x)\cdot x}$

(c) sub $u = \sin x$ $\quad du = \cos x\,dx$

$= \int_{x=0}^{x=\pi/2} u\,du = \frac{u^2}{2}\Big|_{x=0}^{x=\pi/2} = \frac{\sin^2 x}{2}\Big|_0^{\pi/2} = \frac{1}{2}$

(d) split into $\lim_{a\to 0^-}\int_{-1}^{a}\frac{dx}{x^4} + \lim_{b\to 0^+}\int_b^1 \frac{dx}{x^4}$

first compute $\lim_{a\to 0^-}\left(\int_{-1}^{a}\frac{dx}{x^4}\right) = \lim_{a\to 0^-}\left(-\frac{x^{-3}}{3}\Big|_{-1}^{a}\right) = \lim_{a\to 0^-}\left(\frac{-1}{3a^3} - \frac{1}{3}\right)$

diverges

so whole integral diverges
(don't need to check second part of integral now)

(5) (a) $\frac{dy}{dx} + (-1)y = e^x$ Integrating factor $e^{\int -1\,dx} = e^{-x}$

so $e^{-x}y = \int e^x\cdot e^{-x}\,dx = \int 1\,dx = x + C$, so $y = xe^x + Ce^x$

(5) (b) if $y = xe^x + Ce^x$ then $\frac{dy}{dx} = xe^x + e^x + Ce^x$

so $\frac{dy}{dx} - y = (xe^x + e^x + Ce^x) - (xe^x + Ce^x) = e^x$ ✓

(6) (a) Ratio Test: $\lim_{k\to\infty} \left| \frac{3^{k+1} \cdot x^{k+1}}{(k+1)^{1/2}} \cdot \frac{k^{1/2}}{3^k \cdot x^k} \right| = \lim_{k\to\infty} \left| 3x \frac{k^{1/2}}{(k+1)^{1/2}} \right| = |3x|$

so $|3x| < 1 \Rightarrow -\frac{1}{3} < x < \frac{1}{3}$

check endpoints: when $x = -\frac{1}{3}$ $\sum_{k=1}^{\infty} \frac{3^k(-1/3)^k}{\sqrt{k}} = \sum_{k=1}^{\infty} \frac{(-1)^k}{\sqrt{k}}$

converges by Alt. Series test

when $x = \frac{1}{3}$ get $\sum_{k=1}^{\infty} \frac{3^k(\frac{1}{3}^k)}{\sqrt{k}} = \sum_{k=1}^{\infty} \frac{1}{\sqrt{k}} = \sum_{k=1}^{\infty} \frac{1}{k^{1/2}}$

diverges as this is a p-series with $p = \frac{1}{2} < 1$

so Interval of Conv. is $-\frac{1}{3} \leq x < \frac{1}{3}$

(Radius of conv. is $\frac{1}{3}$)

(b) $\lim_{k\to\infty} \left| \frac{(k+1)^2 (x-1)^{k+1}}{2^{k+1}} \cdot \frac{2^k}{k^2 (x-1)^k} \right| = \lim_{k\to\infty} \left| \frac{k^2+2k+1}{k^2} \cdot \frac{(x-1)}{2} \right|$

$= \frac{|x-1|}{2} < 1$

$\Rightarrow -1 < \frac{x-1}{2} < 1$

or $-2 < x - 1 < 2$

or $-1 < x < 3$

check endpoints:

for $x = -1$ $\sum_{k=1}^{\infty} \frac{k^2(-2)^k}{2^k} = \sum_{k=1}^{\infty} (-1)^k \cdot k^2$

here $\lim_{k\to\infty} ((-1)^k \cdot k^2) \neq 0$ so diverges

for $x = 3$ $\sum_{k=1}^{\infty} \frac{k^2 \cdot 2^k}{2^k} = \sum_{k=1}^{\infty} k^2$ also diverges

$\lim_{k\to\infty} k^2 \neq 0$

so Interval of Convergence is $-1 < x \leq 3$

and Radius of convergence is 2

(6) (c) $$\lim_{k\to\infty}\left|\frac{5^{k+1}\cdot(x-\frac{1}{2})^{2k+2}}{(k+3)!}\cdot\frac{(k+2)!}{5^k(x-\frac{1}{2})^{2k}}\right|$$

$$=\lim_{k\to\infty}\left|\frac{5(x-\frac{1}{2})^2}{k+3}\right|=0$$ so converges for all x

radius of conv. is ∞

(7) (a) Rate in − Rate out

Rate in $= 2\,\frac{\text{gal}}{\text{min}}\cdot\frac{1}{100}\frac{\text{lb}}{\text{gal}}=\frac{1}{50}\frac{\text{lb}}{\text{min}}$

Rate out $= \frac{C(t)}{80}\cdot 2\frac{\text{gal}}{\text{min}}=\frac{C}{40}\frac{\text{lb}}{\text{min}}$

so diff eqn. is $\frac{dC}{dt}=\frac{1}{50}-\frac{C}{40}$ with $C(0)=2$

(b) $\frac{dC}{dt}+\frac{1}{40}C=\frac{1}{50}$ Integrating factor is $e^{\int\frac{1}{40}dt}=e^{t/40}$

so $Ce^{t/40}=\frac{1}{50}\int e^{t/40}\,dt=\frac{40}{50}\left(e^{t/40}\right)+k$ (a constant)

or $C(t)=e^{-t/40}\left(\frac{40}{50}e^{t/40}+k\right)$

$=\frac{4}{5}+ke^{-t/40}$

since $C(0)=2$, then $2=\frac{4}{5}+ke^{0}=\frac{4}{5}+k$

so $k=\frac{6}{5}$

Thus $C(t)=\frac{4}{5}+\frac{6}{5}e^{-t/40}$

(c) $\lim_{t\to\infty}\left(\frac{4}{5}+\frac{6}{5}e^{-t/40}\right)=4/5$, so the amount of chlorine approaches $\frac{4}{5}$ lb over time.

(8) (a) $f(x)=(1+x)^{1/4}$

$f'(x)=\frac{1}{4}(1+x)^{-3/4}$

$f''(x)=\frac{-3}{16}(1+x)^{-7/4}$

$f'''(x)=\frac{21}{64}(1+x)^{-11/4}$

$f^{(4)}(x)=\frac{-231}{256}(1+x)^{-15/4}$

so $f(0)=1$, $f'(0)=\frac{1}{4}$, $f''(0)=-3/16$, $f'''(0)=\frac{21}{64}$, $f^{(4)}(0)=\frac{-231}{256}$

(8) (a) continued so $f(x) = 1 + \frac{1}{4}x - \frac{3}{32}x^2 + \frac{21}{64 \cdot 3!}x^3 - \frac{231}{256 \cdot 4!}x^4 + \cdots$

(b) $\cos(x) = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \cdots$

so $\cos(2x) = 1 - \frac{(2x)^2}{2} + \frac{(2x)^4}{4!} - \cdots$

$= 1 - \frac{4x^2}{2} + \frac{16x^4}{24} - \cdots$

so $\sin^2 x = \frac{1}{2} - \frac{1}{2}\left(1 - 2x^2 + \frac{2}{3}x^4 - \cdots\right)$

$= \frac{1}{2} - \frac{1}{2} + x^2 - \frac{1}{3}x^4 = x^2 - \frac{x^4}{3} + \cdots$

(9) (a) $\text{Work} = \int_{5000}^{6000} \frac{k}{r^2}\,dr = \left(-\frac{k}{r}\right)\Big|_{5000}^{6000} = \frac{-k}{6000} + \frac{k}{5000}$

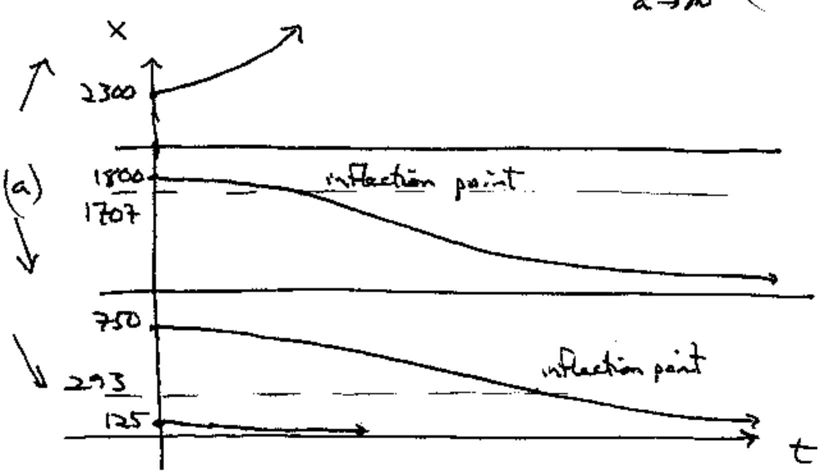
$= \frac{k}{30{,}000}$

(b) $\text{Work} = \int_{6000}^{\infty} \frac{k}{r^2}\,dr = \lim_{a\to\infty} \int_{6000}^{a} \frac{k}{r^2}\,dr$

$= \lim_{a\to\infty}\left(-\frac{k}{r}\Big|_{6000}^{a}\right) = \lim_{a\to\infty}\left(\frac{k}{6000} - \frac{k}{a}\right)$

$= \frac{k}{6000}$

(10) (a)



(b) so for any $x(0) > 2000$ the spaceship will escape