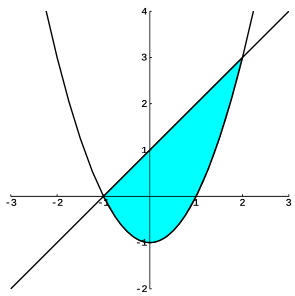
The Definite Integral

1. *The definite integral is defined to be*

   (a) *a limit of Riemann sums.*
   (b) *the difference in the evaluation of an antiderivative at the endpoints of the interval.*
   (c) *a signed area.*
   (d) *all of the above.*

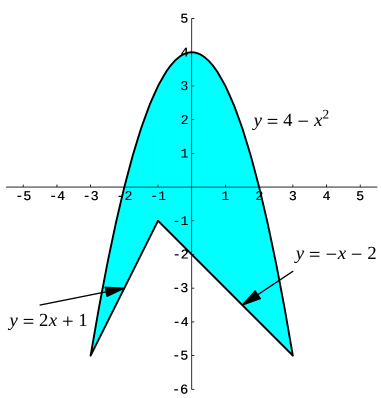
   Solution. $\boxed{\text{(a)}}$. We often *interpret* the definite integral as signed area, and we *compute* the definite integral using (b). However, neither of those is the *definition* of the definite integral.

2. *Write an integral or a sum and/or difference of integrals that gives the area enclosed by the graphs of* $y = x^2 - 1$ *and* $y = x + 1$.

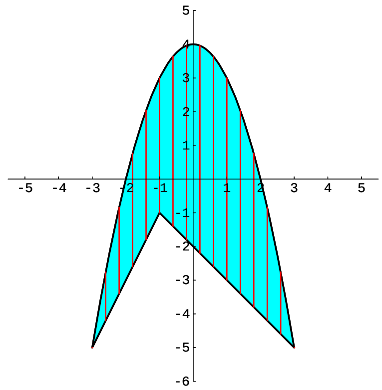


   Solution. The simplest answer is $\boxed{\int_{-1}^{2} [(x+1) - (x^2 - 1)]\ dx}$.

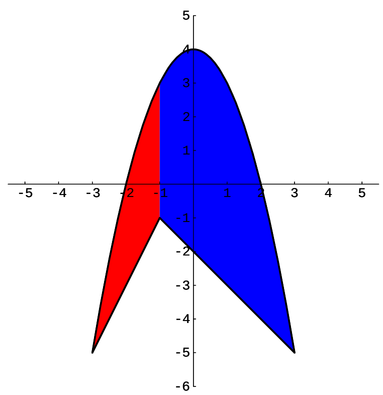
3. *Find the area of the following region.*



   Solution. According to our general theme, we should try to slice, approximate, sum, and take a limit. Let's try slicing vertically:



Notice that there are two "types" of slices in this picture. When x is in the interval $[-3, -1]$, the slices have height $(4-x^2)-(2x+1)$. When x is in the interval $[-1, 3]$, the slices have height $(4-x^2)-(-x-2)$. We should really think about these two cases separately:



We'll find the area of the red piece and the blue piece separately and then just add those together. As we've already said, in the red piece, the slices have height $(4 - x^2) - (2x + 1)$, so the area of the red piece is $\int_{-3}^{-1} [(4 - x^2) - (2x + 1)]\ dx$. In the blue piece, the slices have height $(4 - x^2) - (-x - 2)$, so the area of the blue piece is $\int_{-1}^{3} [(4 - x^2) - (-x - 2)]\ dx$. Thus, the total area is $\boxed{\int_{-3}^{-1} [(4 - x^2) - (2x + 1)]\ dx + \int_{-1}^{3} [(4 - x^2) - (-x - 2)]\ dx}$. We can use the Fundamental Theorem of Calculus to evaluate these integrals, and we get $\boxed{\frac{38}{3}}$.