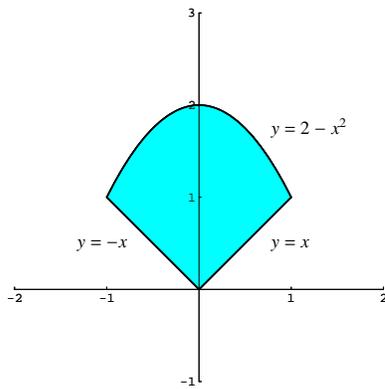1. *Find the area of the region shown. (You may leave your answer as an integral or sum/difference of integrals.)*


Solution. Remember that the basic idea in every integration problem is: slice, approximate, sum, and take a limit. What this boils down to is finding an expression for what's going on with the k-th slice. In this case, we're looking for area, so we want to slice the region and then approximate the area of the k-th slice.

There is a slight hitch though. Let's imagine that we slice the region vertically:
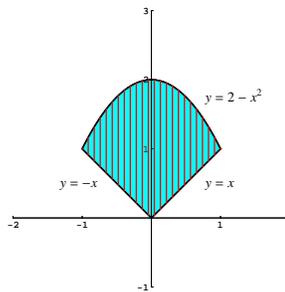
The problem is that we can't write a good description for the k-th slice. If the slice is on the left side of the y-axis, then its height would be about $(2 - x_k^2) - (-x_k)$. If the slice is on the right side, then its height is about $(2 - x_k^2) - x_k$. The solution to this problem is to break the region into two pieces and find their areas separately.
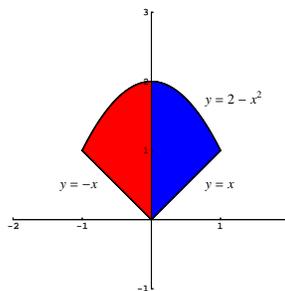
Actually, since the picture is symmetric, we can see right away that the red area is equal to the blue area. So, we can get by just finding the blue area and multiplying that by 2 to get the final answer.

So, let's focus on the right side. As usual, we slice into n pieces of equal width Δx.
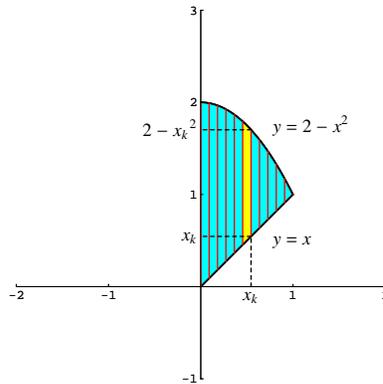
We can pretend that the k-th slice is a rectangle of width Δx and height $(2 - x_k^2) - x_k$, so we end up with
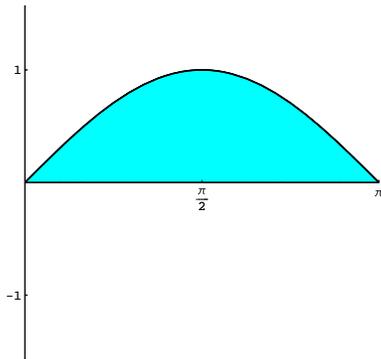
$$\boxed{\text{area of the } k\text{-th slice} \approx [(2 - x_k^2) - x_k]\Delta x}$$

Then we know that the limit of Riemann sums is $\displaystyle\lim_{n\to\infty}\sum_{k=1}^{n}[(2 - x_k^2) - x_k]\Delta x$, which is equal to the integral $\displaystyle\int_0^1 [(2 - x^2) - x]\ dx$.

Finally, remember that this expression only represents half of the original area, so our final answer is
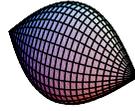
$$\boxed{2\int_0^1 [(2 - x^2) - x]\ dx}.$$

2. Here is one loop of the sine curve.

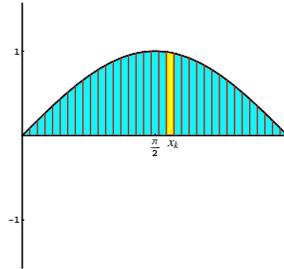

(a) *If you rotate this region about the x-axis, what shape do you get? What is its volume? (You do not need to evaluate your integral.)*

Solution. The solid looks like a football.

To find its volume, we slice and figure out what's happening with the k-th slice.

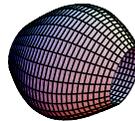The k-th slice is very close to being a disk with radius $\sin x_k$ and thickness Δx, so

$$\boxed{\text{volume of } k\text{-th slice} \approx \pi(\sin x_k)^2 \Delta x}$$

Adding these up and taking the limit gives $\displaystyle\lim_{n\to\infty} \sum_{k=1}^{n} \pi(\sin x_k)^2 \Delta x$, which is equal to the integral

$$\boxed{\int_0^\pi \pi(\sin x)^2 \ dx}.$$

(b) *If you rotate the region about the line $y = -1$, what shape do you get? What is its volume? (You do not need to evaluate your integral.)*

Solution. This looks like a bead.

Again, to find its volume, we slice and figure out what's happening with the k-th slice. Now, the k-th slice isn't a disk any more; it's close to being a washer (disk with a round hole cut out). The outer radius of the washer is $\sin x_k + 1$, the inner radius is 1, and the thickness is Δx. Therefore,

$$\boxed{\text{volume of } k\text{-th slice} \approx \pi[(\sin x_k + 1)^2 - 1^2]\Delta x}$$

Adding these up and taking the limit gives us $\displaystyle\lim_{n\to\infty} \sum_{k=1}^{n} \pi[(\sin x_k + 1)^2 - 1^2]\Delta x$, which is equal to

the integral $\boxed{\int_0^\pi \pi[(\sin x + 1)^2 - 1] \ dx}.$

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