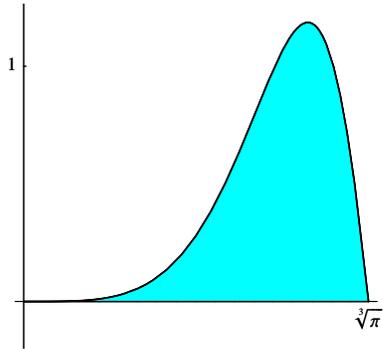
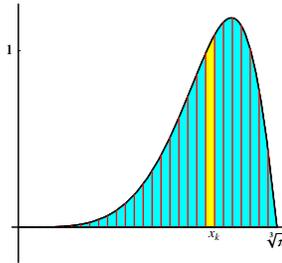


More on Volumes

1. This is the curve $y = x \sin x^3$. If we rotate this region about the y -axis, what is the volume of the resulting solid? (Once you get an integral, try to evaluate it.)



Solution. We may either slice the region horizontally or vertically. Slicing horizontally will give us washers, but to find the inner radius, we would need to solve $y = x \sin x^3$ for x . We don't really know how to do that, so we had better use vertical slices instead.



When we rotate the k -th slice about the y -axis, we get a hollow tube (like a paper towel tube). The radius of this tube is x_k , the height is $x_k \sin x_k^3$, and the thickness is Δx . Therefore, the volume of the tube is approximately $2\pi x_k(x_k \sin x_k^3)\Delta x = 2\pi x_k^2 \sin x_k^3 \Delta x$.

To approximate the volume of the whole solid, we have to sum up the volumes of all of the slices (slice 1 through slice n), which gives $\sum_{k=1}^n 2\pi x_k^2 \sin x_k^3 \Delta x$. To make the approximation accurate, we need to

use more and more slices, so we take the limit as $n \rightarrow \infty$. So, our answer is $\lim_{n \rightarrow \infty} \sum_{k=1}^n 2\pi x_k^2 \sin x_k^3 \Delta x$,

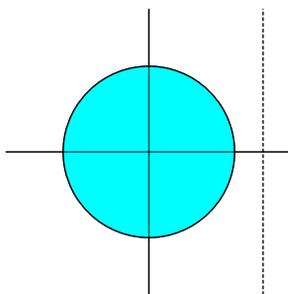
which is the integral $\int_0^{\sqrt[3]{\pi}} 2\pi x^2 \sin x^3 dx$. We can evaluate this integral using substitution. Let $u = x^3$.

Then, $du = 3x^2 dx$. Since x goes from 0 to $\sqrt[3]{\pi}$, u goes from 0 to π . So,

$$\begin{aligned} \int_0^{\sqrt[3]{\pi}} 2\pi x^2 \sin x^3 dx &= \int_0^{\pi} 2\pi \sin u \frac{du}{3} \\ &= \frac{2\pi}{3} \int_0^{\pi} \sin u du \\ &= \frac{2\pi}{3} (-\cos u) \Big|_0^{\pi} \\ &= \frac{2\pi}{3} (-\cos \pi + \cos 0) \\ &= \boxed{\frac{4\pi}{3}} \end{aligned}$$

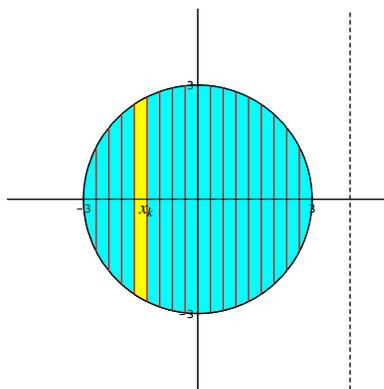
2. How can you describe a bagel as a solid of revolution? (That is, what sort of region would you rotate, and what line would you rotate it about?)

Solution. We can get a bagel by rotating a disk around a line, something like this (rotate the disk around the dotted line):



3. The disk of radius 3 centered at the origin is rotated about the line $x = 4$. Find the volume using vertical slices. (It is also possible to do it using horizontal slices, and you might want to try that for extra practice.)

Solution. Here are the slices.



The equation of the circle is $x^2 + y^2 = 9$, or $y = \pm\sqrt{9 - x^2}$. $y = \sqrt{9 - x^2}$ is the equation of the top half of the circle, and $y = -\sqrt{9 - x^2}$ is the equation of the bottom half of the circle.

Rotating the k -th slice gives (approximately) a paper towel tube with thickness Δx , radius $4 - x_k$, and height $2\sqrt{9 - x_k^2}$. So, the volume of the k -th piece is approximately $2\pi(4 - x_k)(2\sqrt{9 - x_k^2})\Delta x = 4\pi(4 - x_k)\sqrt{9 - x_k^2}\Delta x$.

Adding these up and taking the limit, we get the integral $\int_{-3}^3 4\pi(4 - x)\sqrt{9 - x^2} dx = 4\pi \int_{-3}^3 (4 - x)\sqrt{9 - x^2} dx$. To evaluate this integral, let's first multiply it out a little:

$$\begin{aligned} 4\pi \int_{-3}^3 (4 - x)\sqrt{9 - x^2} dx &= 4\pi \left[\int_{-3}^3 4\sqrt{9 - x^2} dx - \int_{-3}^3 -x\sqrt{9 - x^2} dx \right] \\ &= 16\pi \int_{-3}^3 \sqrt{9 - x^2} dx + 4\pi \int_{-3}^3 x\sqrt{9 - x^2} dx \end{aligned}$$

Now, we have two integrals to evaluate. We don't know an antiderivative of $\sqrt{9 - x^2}$, but $y = \sqrt{9 - x^2}$ is just the graph of the top half of the circle in our picture. So, $\int_{-3}^3 \sqrt{9 - x^2} dx$ is the area of the top half of the circle, and we know that a circle of radius 3 has area 9π . So, $\int_{-3}^3 \sqrt{9 - x^2} dx = \frac{9\pi}{2}$.

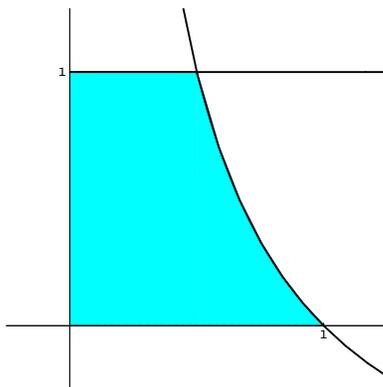
The second integral $\int_{-3}^3 x\sqrt{9 - x^2} dx$ can be done using substitution: let $u = 9 - x^2$. Then $du = -2x dx$. So, we can rewrite the integrand as $\int -\sqrt{u}\frac{du}{2}$. Since x goes from -3 to 3 , u goes from 0 to 0 , and $\int_0^0 -\frac{\sqrt{u}}{2} du = 0$.

So, our final answer is $16\pi \cdot \frac{9\pi}{2} = \boxed{72\pi^2}$.

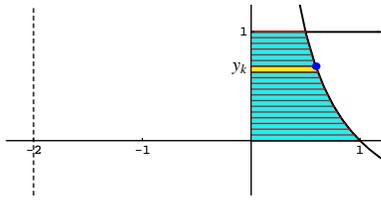
4. Let \mathcal{R} be the region enclosed by the x -axis, the y -axis, $y = 1$, and $y = \frac{1}{x} - 1$.

- (a) Find the volume generated when \mathcal{R} is rotated about the line $x = -2$.

Solution. Here is the region \mathcal{R} .



As always, we should first decide whether we want to use vertical or horizontal slices. In this case, if we use vertical slices, we will have to split the region up into where $x < \frac{1}{2}$ and $x > \frac{1}{2}$ because the slices will have different descriptions. Instead, let's use horizontal slices.



Since we're using horizontal slices, we should describe things in terms of y , so let's solve $y = \frac{1}{x} - 1$ for x in terms of y .

$$\begin{aligned} y &= \frac{1}{x} - 1 \\ y + 1 &= \frac{1}{x} \\ \frac{1}{y + 1} &= x \end{aligned}$$

So, the blue point is $\left(\frac{1}{y_k + 1}, y_k\right)$.

After we rotate our horizontal slice, we will end up with something that is approximately a washer (or CD) with inner radius 2, outer radius $\frac{1}{y_k + 1} + 2$, and thickness Δy . So, the volume of the k -th slice is approximately $\left[\pi \left(\frac{1}{y_k + 1} + 2\right)^2 - \pi \cdot 2^2\right] \Delta y$. Adding these up and taking the limit gives

the integral $\int_0^1 \left[\pi \left(\frac{1}{y + 1} + 2\right)^2 - \pi \cdot 2^2\right] dy$. To compute this, we'll first simplify:

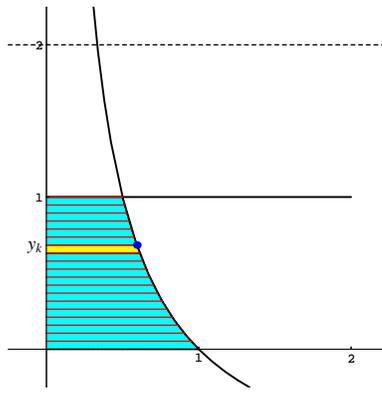
$$\begin{aligned} \int_0^1 \left[\pi \left(\frac{1}{y + 1} + 2\right)^2 - \pi \cdot 2^2\right] dy &= \pi \int_0^1 \left[\left(\frac{1}{y + 1} + 2\right)^2 - 2^2\right] dy \\ &= \pi \int_0^1 \left[\frac{1}{(y + 1)^2} + \frac{4}{y + 1} + 4 - 4\right] dy \\ &= \pi \int_0^1 \left[\frac{1}{(y + 1)^2} + \frac{4}{y + 1}\right] dy \end{aligned}$$

To evaluate, we'll use the substitution $u = y + 1$. Then, $du = dy$ and u goes from 1 to 2 (since y went from 0 to 1). So, we have

$$\begin{aligned} \text{volume} &= \pi \int_1^2 \left(\frac{1}{u^2} + \frac{4}{u}\right) du \\ &= \pi \left(-u^{-1} + 4 \ln |u|\right) \Big|_1^2 \\ &= \pi \left[\left(-\frac{1}{2} + 4 \ln 2\right) - (-1)\right] \\ &= \boxed{\pi \left(\frac{1}{2} + 4 \ln 2\right)} \end{aligned}$$

(b) Find the volume generated when \mathcal{R} is rotated about the line $y = 2$.

Solution. For the same reason as in part (a), we'll use horizontal slices.



As we figured out in part (a), the blue point has coordinates $\left(\frac{1}{y_k+1}, y_k\right)$.

After we rotate our horizontal slice, we will end up with a paper towel tube with radius $2 - y_k$, height $\frac{1}{y_k+1}$, and thickness Δy . So, the volume of this slice is approximately $2\pi(2 - y_k)\frac{1}{y_k+1}\Delta y = 2\pi\frac{2-y_k}{y_k+1}$. Adding these up and taking the limit gives $\lim_{n \rightarrow \infty} \sum_{k=1}^n 2\pi\frac{2-y_k}{y_k+1}$, which is just the integral

$$\int_0^1 2\pi\frac{2-y}{y+1} dy.$$

Now, we have to actually evaluate the integral. Let's try using $u = y + 1$. Then, $du = dy$. Also, $y = u - 1$, so $2 - y = 2 - (u - 1) = 3 - u$. Since y goes from 0 to 1, u goes from 1 to 2. So,

$$\begin{aligned} \int_0^1 2\pi\frac{2-y}{y+1} dy &= 2\pi \int_0^1 \frac{2-y}{y+1} dy \\ &= 2\pi \int_1^2 \frac{3-u}{u} du \\ &= 2\pi \int_1^2 \left(\frac{3}{u} - 1\right) du \\ &= 2\pi (3 \ln|u| - u)\Big|_1^2 \\ &= 2\pi[(3 \ln 2 - 2) - (3 \ln 1 - 1)] \\ &= \boxed{2\pi(3 \ln 2 - 1)} \end{aligned}$$