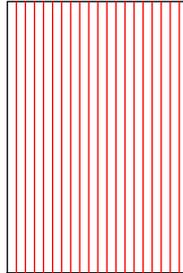


Density and Slicing

1. A seaside village, Playa del Carmen, is in the shape of a rectangle 4 miles wide and 6 miles long. The sea lies along a 6-mile long side. People prefer to live near the water, so the density of people is given by $\rho(x) = 10000 - 800x$ people per square mile, where x is the distance from the seaside. We would like to find the population of the village.

(a) *Show in a sketch how to slice up the region.*

Solution. We'll slice the region parallel to the sea, like this:



The reason we do it this way is in part (c).

(b) *What is the area of the k -th slice?*

Solution. We'll call the width of each slice Δx . Then, the area of the k -th slice is $6\Delta x$.

(c) *What is the approximate population in the k -th slice?*

Solution. This is really the key to the problem. Because of the way we sliced, the population density within each slice is close to being constant. Therefore, we can approximate the population as (population density) times (area), or $\rho(x_k) \cdot 6\Delta x = 6\rho(x_k)\Delta x$.

(d) *Write a general Riemann sum to estimate the total population of the city.*

Solution. To approximate the total population, we just add up the approximate population in all slices, which gives $\sum_{k=1}^n 6\rho(x_k)\Delta x$.

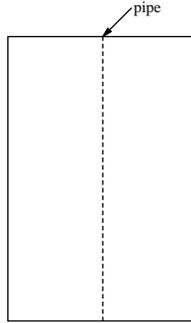
(e) *Find a definite integral expressing the population of the village.*

Solution. The assumption we made in our approximation was that the population density in each slice was constant. This assumption becomes more accurate as we use more slices, since each slice will be thinner. Therefore, to get the actual population, we should take the limit as $n \rightarrow \infty$.

We know that $\lim_{n \rightarrow \infty} \sum_{k=1}^n 6\rho(x_k)\Delta x$ is the same as the integral $\int_0^4 6\rho(x) dx$.

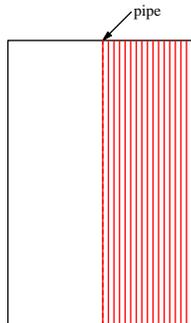
2. A rectangular plot of farm land is 300 meters by 200 meters. A straight irrigation pipe 300 meters long runs down the center of the plot, dividing it in half lengthwise. The farmer's yield decreases as the distance from the irrigation pipe increases. Suppose that the yield is given by $\rho(x)$ grams per square meter, where x is the distance in meters from the irrigation pipe. Write an integral giving the total yield from the plot.

Solution. Here's a sketch.



First, because of symmetry, the yield in the left half of the plot is the same as the yield in the right half. Therefore, we only need to figure out the yield in one half; let's do the right half.

Since the yield depends on the distance from the irrigation pipe, we want to slice in which the distance to the irrigation pipe is approximately constant. Therefore, we'll slice parallel to the pipe:



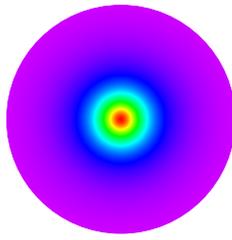
Since x represents the distance to the pipe, the leftmost slice starts where $x = 0$, and the rightmost slice ends where $x = 100$.

The area of each slice is $300\Delta x$. In the k -th slice, the yield rate is approximately $\rho(x_k)$ grams per square meter, so the yield is approximately $\rho(x_k) \cdot 300\Delta x = 300\rho(x_k)\Delta x$. Summing over all n slices and taking the limit as $n \rightarrow \infty$ gives us $\int_0^{100} 300\rho(x) dx$ as the yield for half of the plot. Therefore,

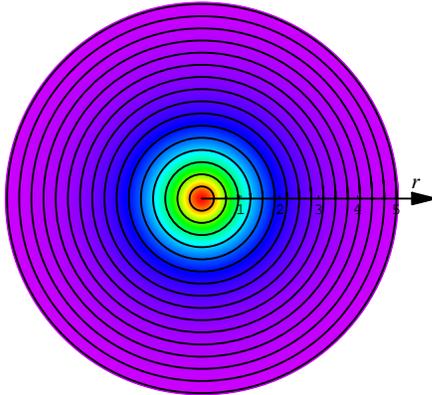
the total yield is $2 \int_0^{100} 300\rho(x) dx$.

3. *People in the Boston area like to live near the city center, so the population density around Boston is $\rho(r) = \frac{36,000}{r^2+2r+1}$ people per square mile, where r is the distance in miles to the center of Boston. Find the number of people who live within 5 miles of the center of Boston.*

Solution. The region we are talking about (within 5 miles of the center of Boston) is a disk, with the highest population density at the center. Here is a plot showing the population density in the city (red represents the highest density, and purple represents the lowest):



Just like in the other problems, we want to slice the region so that the population density in each slice is almost constant. In this case, we can accomplish that by slicing into concentric rings.



The population density in the k -th slice is approximately equal to $\rho(r_k)$. The area of slice k is approximately $2\pi r_k \Delta r$. So, the number of people living in the k -th slice is approximately $\rho(r_k) \cdot 2\pi r_k \Delta r = 2\pi r_k \rho(r) \Delta r$. Summing over all n slices and taking the limit as $n \rightarrow \infty$ gives the integral

$$\int_0^5 2\pi r \rho(r) dr.$$