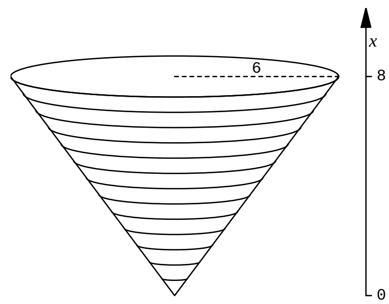
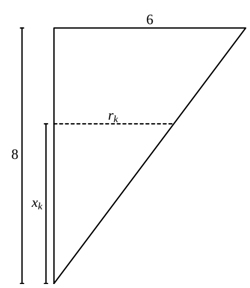
1. *A cone with height 8 inches and radius 6 inches is filled with flavored slush. When the cup is held upright with the pointed end resting on a table, the density of flavoring syrup in the cup varies with height above the table. Suppose $\rho(x)$ gives the number of ounces of syrup per cubic inch, where x is the distance from the table top. Write an integral giving the total amount of syrup in the cup.*

   Solution. Since the density varies with height, we will slice the cone like this:



   The k-th slice looks approximately like a disk of thickness Δx. To approximate the volume of the slice, we need to find its radius. Moreover, we should do this in terms of x_k since we're going to want to end up with an integral in terms of x.

   To find the radius r_k of the k-th slice in terms of x_k, we use similar triangles:



   This shows us that $\frac{6}{8} = \frac{r_k}{x_k}$, so $r_k = \frac{6}{8}x_k = \frac{3}{4}x_k$. Therefore, the volume of the k-th slice is approximately $\pi\left(\frac{3}{4}x_k\right)^2 \Delta x = \frac{9\pi}{16}x_k^2\Delta x$ cubic inches. The amount of syrup in the k-th slice is approximately $[\rho(x_k)\text{ounces per cubic inch}] \cdot \left[\frac{9\pi}{16}x_k^2\Delta x\text{cubic inches}\right] = \frac{9\pi}{16}x_k^2\rho(x_k)\Delta x$ ounces. Summing the amount in each slice gives the Riemann sum approximation $\displaystyle\sum_{k=1}^{n}\frac{9\pi}{16}x_k^2\rho(x_k)\Delta x$. Taking the limit as $n \to \infty$ gives the integral $\boxed{\displaystyle\int_0^8 \frac{9\pi}{16}x^2\rho(x)\,dx}$.

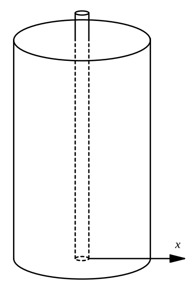
2. *Suppose the density of a planet is given by the function $\rho(r) = \dfrac{40000}{1 + 0.0001r^3}$ kilograms per cubic kilometer, where r is the distance in kilometers from the center of the planet. Find the total mass of the planet if its radius is 8000 km. (You do not need to evaluate your integral.)*

Solution. Since the density varies with the distance to the center, we should slice into concentric spherical shells. Each shell will have a small thickness Δr, and a good approximation for the volume of such a spherical shell is its surface area multiplied by the thickness Δr.

The k-th slice has outer radius r_k and inner radius r_{k-1}, so its volume is approximately $4\pi r_k^2 \Delta r$ km^3. The mass of this slice is approximately $[\rho(r_k)\text{kg} / \text{km}^3] \cdot [4\pi r_k^2 \Delta r \text{ km}^3] = 4\pi r_k^2 \rho(r_k)\Delta r$ kg. Adding these up gives the Riemann sum approximation $\sum_{k=1}^{n} 4\pi r_k^2 \rho(r_k)\Delta r$. Taking the limit gives the integral $\boxed{\int_0^{8000} 4\pi r^2 \rho(r)\, dr}$. If we wanted a numeric answer, we could integrate using the substitution $u = 1 + 0.0001r^3$.

3. *A cylindrical candle of height* 50 *mm and radius* 12 *mm is formed by repeatedly dipping a wick of radius* 1 *mm into hot wax and then allowing the new layer of wax to dry. The density of each new layer is slightly different, so the density of the candle varies with the distance to the wick. If $\rho(x)$ gives the density in grams per cubic mm of the wax, where x measures the distance to the wick, write an integral giving the mass of the candle.*

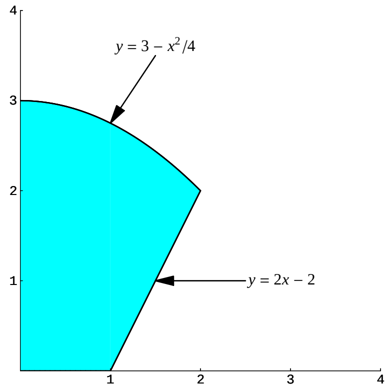
Solution. Here is a (crude) sketch of our candle.



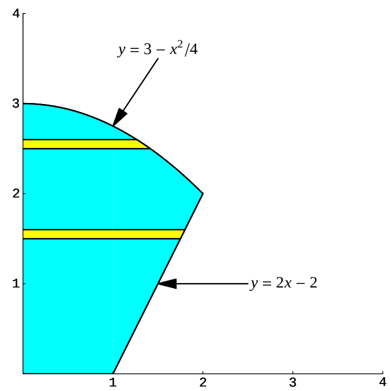
We should slice using cylindrical shells (also known as paper towel tubes, just like what we used in some of the volumes of revolution problems).

Each cylindrical shell has height 50 mm and thickness Δx. The k-th shell is distance x_k from the wick, so its radius is $x_k + 1$ (because we must take into account the radius of the wick). So, the volume of the k-th shell is approximately $2\pi(x_k + 1) \cdot 50 \cdot \Delta x = 100\pi(x_k + 1)\Delta x$ cubic inches. Then, its mass is approximately $[\rho(x_k)$ grams per cubic mm$] \cdot [100\pi(x_k + 1)\Delta x$ cubic mm$] = 100\pi(x_k + 1)\rho(x_k)\Delta x$ grams. Adding these up gives the Riemann sum approximation $\sum_{k=1}^{n} 100\pi(x_k + 1)\rho(x_k)\Delta x$. Taking the limit as $n \to \infty$ gives the definite integral $\boxed{\int_0^{11} 100\pi(x + 1)\rho(x)\, dx}$.

4. *We can model a muffin as a solid of revolution, obtained by rotating the following region about the y-axis. Due to a poor recipe, the chocolate chips in our muffin tend to sink to the bottom. The amount of chocolate in the muffin is given by $\rho(y) = 5 - y$ grams per cubic inch, where y represents the distance to the bottom of the muffin. Find the total amount of chocolate in the muffin.*

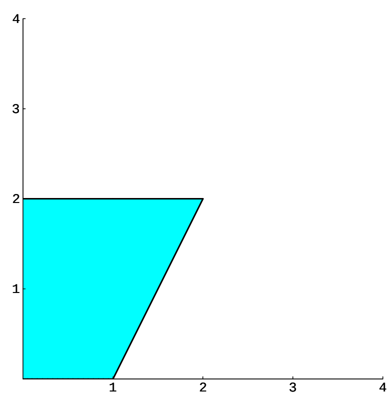


Solution. Since the chocolate density varies with the distance to the bottom of the muffin, we must slice parallel to the bottom of the muffin. Each slice is approximately a disk. Here are two representative slices:



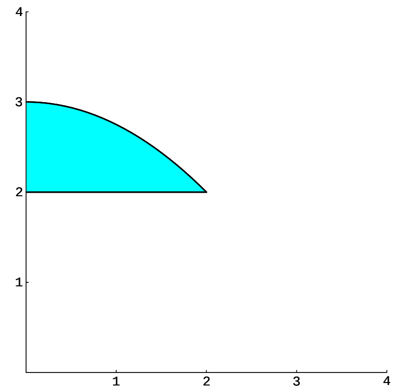
Notice that these two slices have different descriptions: the top one should be described using the curve $y = 3 - \frac{x^2}{4}$, while the bottom one should be described using the curve $y = 2x - 2$. So, we should really consider the top and bottom parts of the muffin separately.

Let's first focus on the bottom part of the muffin, which we get by rotating this region:



Here, the slices are disks, with the radius being the horizontal distance between the y-axis and $y = 2x - 2$. To find this horizontal distance, we need to solve $y = 2x - 2$ for x, which gives $x = \frac{y+2}{2}$. Therefore, the radius of the k-th slice is $\frac{y_k+2}{2}$. This means that its volume is approximately $\pi\left(\frac{y_k+2}{2}\right)^2 \Delta y$, so the amount of chocolate in this slice is approximately $\rho(y_k)\cdot\pi\left(\frac{y_k+2}{2}\right)^2 \Delta y = \pi\left(\frac{y_k+2}{2}\right)^2 \rho(y_k)\Delta y$. Adding these up and taking the limit gives us the integral $\int_0^2 \pi\left(\frac{y+2}{2}\right)^2 \rho(y)\, dy$ for the amount of chocolate in the bottom part of the muffin.

To deal with the top part of the muffin, we use exactly the same reasoning. We get the top part of the muffin by rotating this region:



We need to solve $y = 3 - \frac{x^2}{4}$ for x:

$$\begin{aligned} y &= 3 - \frac{x^2}{4} \\ \frac{x^2}{4} &= 3 - y \\ x^2 &= 4(3-y) \\ x &= 2\sqrt{3-y} \end{aligned}$$

So, the k-th slice is approximately a disk of radius $2\sqrt{3-y_k}$ and thickness Δy, which means its volume is approximately $\pi(2\sqrt{3-y_k})^2\Delta y = 4\pi(3-y_k)\Delta y$. Therefore, the amount of chocolate in this slice is approximately $4\pi(3-y_k)\rho(y_k)\Delta y$. Adding these up and taking the limit gives us the integral $\int_2^3 4\pi(3-y)\rho(y)\, dy$ for the amount of chocolate in the top part of the muffin.

So, the total amount of chocolate in the muffin is $\boxed{\int_0^2 \pi\left(\frac{y+2}{2}\right)^2 \rho(y)\, dy + \int_2^3 4\pi(3-y)\rho(y)\, dy.}$

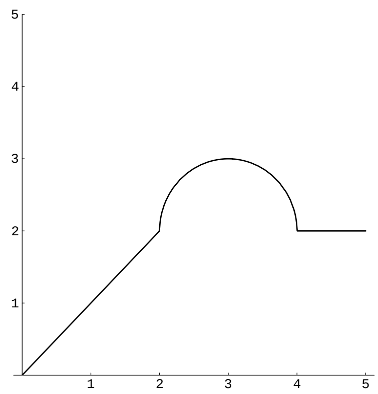
5. *You ride your bike with velocity $v(t) = 3t^2 + 4t - 5$ in the time interval $[0, 3]$. What is your average velocity?*

Solution. It is $\boxed{\frac{1}{3-0}\int_0^3 v(t)\, dt} = \frac{1}{3}\int_0^3 (3t^2+4t-5)\, dt = \frac{1}{3}(t^3+2t^2-5t)\Big|_0^3 = \boxed{10}$.

6. *The temperature outside is given by the function $f(t)$, where t represents the time since 10:00 am. How would you find the average temperature between noon and 5:00 pm?*

Solution. We want the average temperature for the interval $[2, 7]$, and that's $\boxed{\dfrac{1}{7-2}\displaystyle\int_2^7 f(t)\,dt}$.

7. *The graph of a function f is shown. The graph is made up of lines and semicircles. Find the average value of f on the interval* $[1, 5]$.



Solution. The average value of f is $\dfrac{1}{4}\displaystyle\int_1^5 f(t)\,dt$. We know that $\int_1^5 f(t)\,dt$ is the signed area of f from $t = 1$ to $t = 5$, and from the graph, we can see that this signed area is $\frac{9+\pi}{2}$. Therefore, the average value of the function is $\boxed{\frac{9+\pi}{8}}$.