Arc Length and Improper Integrals

1. *Write an integral that gives the length of one arch of the sine curve (so from $x = 0$ to $x = \pi$).*

   Solution. Our formula tells us that it is $\int_0^{\pi} \sqrt{1+\cos^2 x}\, dx$.

2. (a) *Does* $\int_1^{\infty} \frac{1}{x^2}\, dx$ *converge or diverge? If it converges, evaluate it.*

   Solution. We know that $\int_1^{\infty} \frac{1}{x^2}\, dx$ really means $\lim_{b\to\infty} \int_1^{b} \frac{1}{x^2}\, dx$. We can evaluate $\int_1^{b} \frac{1}{x^2}\, dx$ pretty easily: it is $-\frac{1}{x}\Big|_1^b = -\frac{1}{b} + 1 = 1 - \frac{1}{b}$. So, $\int_1^{\infty} \frac{1}{x^2}\, dx = \lim_{b\to\infty} 1 - \frac{1}{b} = \boxed{1}$.

   (b) *Does* $\int_1^{\infty} \frac{1}{x}\, dx$ *converge or diverge? If it converges, evaluate it.*

   Solution. We know that $\int_1^{\infty} \frac{1}{x}\, dx$ really means $\lim_{b\to\infty} \int_1^{b} \frac{1}{x}\, dx$, so

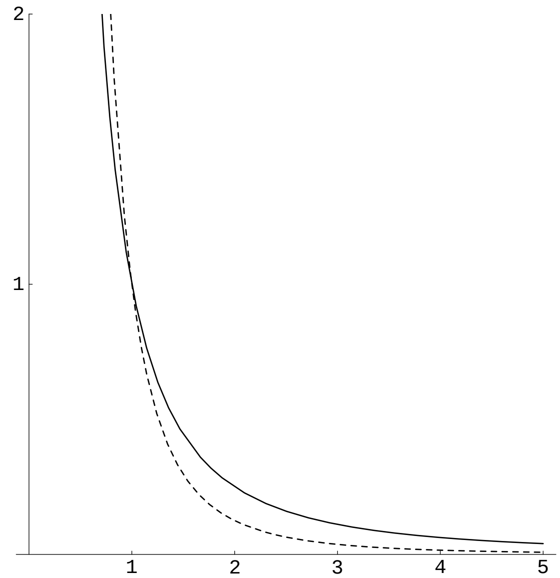
$$\begin{aligned} \int_1^{\infty} \frac{1}{x}\, dx &= \lim_{b\to\infty} \left(\ln|x|\Big|_1^b \right) \\ &= \lim_{b\to\infty} (\ln|b| - \ln 1) \\ &= \lim_{b\to\infty} \ln|b| \text{ since } \ln 1 = 0 \end{aligned}$$

   But we know that $\lim_{b\to\infty} \ln|b| = \infty$, which is a form of diverging, so the improper integral $\int_1^{\infty} \frac{1}{x}\, dx$ diverges.

3. Using #2, can you conclude anything about whether the following integrals converge or diverge? (Try to figure this out without evaluating the integrals!)

   (a) $\int_1^{\infty} \frac{1}{x^3}\, dx$*?*

   Solution. Let's graph $\frac{1}{x^2}$ (the solid curve) and $\frac{1}{x^3}$ (the dashed curve):

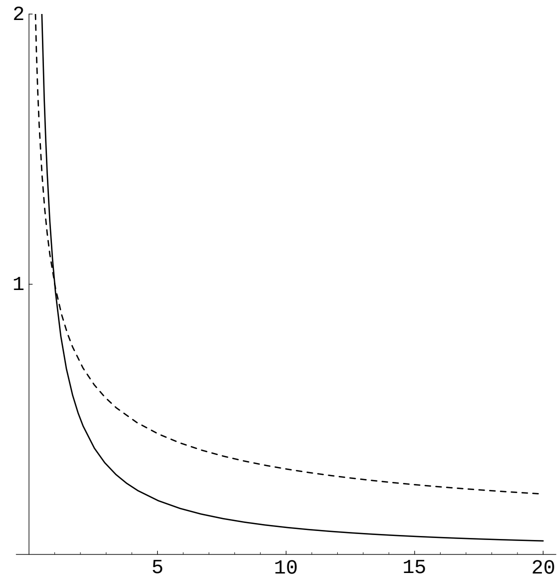


By 2(a), we know that $\int_1^\infty \frac{1}{x^2}\,dx = 1$. Graphically, we interpret this as the area under the curve $y = \frac{1}{x^2}$ to the right of $x = 1$. It is apparent from our picture that the area under $y = \frac{1}{x^3}$ to the right of $x = 1$ should be less than that, so we expect the integral $\int_1^\infty \frac{1}{x^3}\,dx$ to converge. In fact, it does:

$$\begin{aligned}
\int_1^\infty \frac{1}{x^3}\,dx &= \lim_{b\to\infty} \int_1^b \frac{1}{x^3}\,dx \\
&= \lim_{b\to\infty} \left(-\frac{1}{2}x^{-2}\Big|_1^b \right) \\
&= \lim_{b\to\infty} \left(-\frac{1}{2b^2} + \frac{1}{2} \right) \\
&= \frac{1}{2}
\end{aligned}$$

(b) $\int_1^\infty \frac{1}{x^{1/2}}\,dx$?

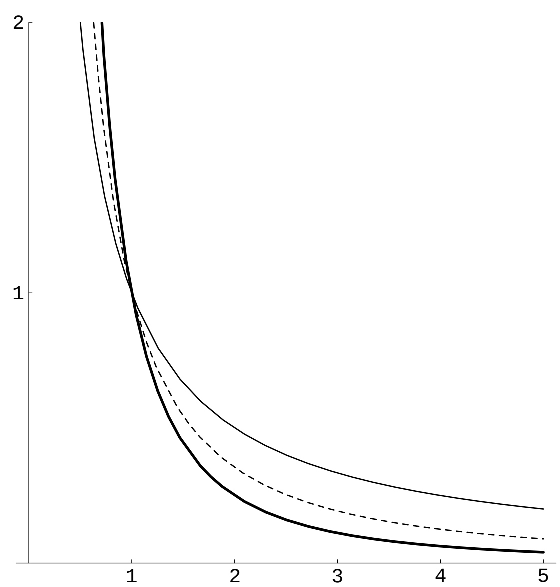
Solution. Let's graph $\frac{1}{x}$ (the solid curve) and $\frac{1}{x^{1/2}}$ (the dashed curve):



Since the graph of $\frac{1}{x^{1/2}}$ is higher than the graph of $\frac{1}{x}$ when $x \geq 1$, the area under $\frac{1}{x^{1/2}}$ to the right of $x = 1$ should be at least as big as the area under $\frac{1}{x}$ to the right of $x = 1$. The area under $\frac{1}{x}$ to the right of $x = 1$ was already infinite, so we expect $\displaystyle\int_1^\infty \frac{1}{x^{1/2}}\, dx$ to diverge.

(c) $\displaystyle\int_1^\infty \frac{1}{x^{3/2}}\, dx$*?*

Solution. The graph of $\frac{1}{x^{3/2}}$ lies between the graphs of $\frac{1}{x}$ and $\frac{1}{x^2}$:



Here, the thin solid graph is $y = \frac{1}{x}$, the thick solid graph is $y = \frac{1}{x^2}$, and the dashed graph is $y = \frac{1}{x^{3/2}}$. Thinking in terms of areas, we can guess that $\displaystyle\int_1^\infty \frac{1}{x^2}\, dx \leq \int_1^\infty \frac{1}{x^{3/2}}\, dx \leq \int_1^\infty \frac{1}{x}\, dx$.

Using our result from #2, this tells us that $1 \leq \int_1^\infty \frac{1}{x^{3/2}}\ dx \leq \infty$. Unfortunately, that doesn't give us enough information to determine whether $\int_1^\infty \frac{1}{x^{3/2}}\ dx$ converges.

In this case, comparing to $\frac{1}{x}$ and $\frac{1}{x^2}$ doesn't help, so we'll evaluate:

$$
\begin{aligned}
\int_1^\infty \frac{1}{x^{3/2}}\ dx &= \lim_{b\to\infty} \int_1^b \frac{1}{x^{3/2}}\ dx \\
&= \lim_{b\to\infty} \left(-2x^{-1/2}\Big|_1^b \right) \\
&= \lim_{b\to\infty} -2b^{-1/2} + 2 \\
&= \lim_{b\to\infty} 2 - \frac{2}{\sqrt{b}} \\
&= 2
\end{aligned}
$$

So, the improper integral converges to 2.