Improper Integrals

Determine whether the following integrals converge or diverge. Explain your reasoning.

1. $\displaystyle\int_{-2}^{2} \frac{x}{x^2-1}\,dx.$

   Solution. The integrand is discontinuous at $x = \pm 1$, so we know we need to split the integral. The improprieties are at -1 and 1, and each of our pieces should have at most one impropriety. So, let's split like this:

   $$\int_{-2}^{2} \frac{x}{x^2-1}\,dx = \int_{-2}^{-1} \frac{x}{x^2-1}\,dx + \int_{-1}^{0} \frac{x}{x^2-1}\,dx + \int_{0}^{1} \frac{x}{x^2-1}\,dx + \int_{1}^{2} \frac{x}{x^2-1}\,dx.$$

   (You could choose a number other than 0 between -1 and 1.)

   Now, we have to evaluate each of the integrals on the right (and they are all improper). Let's first find an antiderivative of $\frac{x}{x^2-1}$ by substituting $u = x^2 - 1$:

   $$\int \frac{x}{x^2-1}\,dx = \frac{1}{2}\int \frac{1}{u}\,du = \frac{1}{2}\ln|u| = \frac{1}{2}\ln|x^2-1|.$$

   Now, we start evaluating our four improper integrals using limits.

   $$\begin{aligned}\int_{-2}^{-1} \frac{x}{x^2-1}\,dx &= \lim_{b\to -1^-} \int_{-2}^{b} \frac{x}{x^2-1}\,dx \\ &= \lim_{b\to -1^-} \frac{1}{2}\ln|x^2-1|\Big|_{-2}^{b} \\ &= \lim_{b\to -1^-} \left(\frac{1}{2}\ln|b^2-1| - \frac{1}{2}\ln 3\right)\end{aligned}$$

   As $b \to -1^-$, $b^2 - 1 \to 0$, so $\ln|b^2-1| \to -\infty$. Thus, this integral diverges.

   Since one of our four pieces diverges, we don't need to bother calculating the other pieces; we already know that the whole integral $\boxed{\text{diverges}}$.

2. $\displaystyle\int_{1}^{\infty} \frac{1}{x^4+2}\,dx.$

   Solution. The integrand here is very similar to $\frac{1}{x^4}$, and we know $\displaystyle\int_1^\infty \frac{1}{x^4}\,dx$ converges. This suggests that we use the Comparison Theorem.

   Notice that $0 \le \frac{1}{x^4+2} \le \frac{1}{x^4}$ for all x. Since $\displaystyle\int_1^\infty \frac{1}{x^4}\,dx$ converges, the Comparison Theorem tells us that $\displaystyle\int_1^\infty \frac{1}{x^4+2}\,dx$ also $\boxed{\text{converges}}$.

3. $\displaystyle\int_{0}^{\infty} \frac{1}{e^x+x}\,dx.$

Solution. Since we don't know how to find an antiderivative of $\frac{1}{e^x+x}$, we should use the Comparison Theorem. Since $0 \leq \frac{1}{e^x+x} \leq \frac{1}{e^x}$ for all x and you saw on your homework that $\displaystyle\int_0^\infty \frac{1}{e^x}\,dx$ converges, the Comparison Theorem tells us that $\displaystyle\int_0^\infty \frac{1}{e^x+x}\,dx$ also $\boxed{\text{converges}}$.

4. $\displaystyle\int_{-\infty}^{\infty} \sin x\;dx.$

Solution. We need to split up this integral because it has two improprieties: the $-\infty$ and the ∞. It doesn't really matter where we split it, so let's split it at 0: $\displaystyle\int_{-\infty}^{\infty} \sin x\;dx = \int_{-\infty}^{0} \sin x\;dx + \int_0^\infty \sin x\;dx.$ Let's do $\displaystyle\int_0^\infty \sin x\;dx$ first. By definition, this is $\displaystyle\lim_{b\to\infty}\int_0^b \sin x\;dx = \lim_{b\to\infty} -\cos x|_0^b = \lim_{b\to\infty}(-\cos b+1)$, which does not exist. Since this piece diverges, we know that the whole integral $\boxed{\text{diverges}}$.

5. $\displaystyle\int_1^\infty \frac{1+e^{-x}}{x}\;dx.$

Solution. We can use the Comparison Theorem: $\frac{1+e^{-x}}{x} \geq \frac{1}{x} \geq 0$ when $x \geq 1$. Since we know that $\displaystyle\int_1^\infty \frac{1}{x}\,dx$ diverges, $\displaystyle\int_1^\infty \frac{1+e^{-x}}{x}\,dx$ must $\boxed{\text{diverge}}$ as well.

6. $\displaystyle\int_1^\infty \frac{\cos^2 x}{x^2}\;dx.$

Solution. We can use the Comparison Theorem: $\frac{1}{x^2} \geq \frac{\cos^2 x}{x^2} \geq 0$. Since we know that $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx$ converges, $\displaystyle\int_1^\infty \frac{\cos^2 x}{x^2}\,dx$ $\boxed{\text{converges}}$ as well.