Probability

Waiting times, shelf-lifes, and equipment failure times are often modeled by exponentially decreasing probability density functions.

1. Suppose $f(t) = 0$ for $t < 0$ and $f(t) = 0.5e^{-ct}$ for $t \geq 0$ is the probability density function for the lifetime of a particular toy (t in years).

   (a) *For what value of c is this a probability density function?*

   Solution. In order for f to be a probability density function, it must satisfy $\int_{-\infty}^{\infty} f(t)\, dt = 1$, or $\int_{-\infty}^{0} f(t)\, dt + \int_{0}^{\infty} f(t)\, dt = 1$. The first integral is 0 since $f(t) = 0$ for $t < 0$.

   If $c = 0$, then $f(t) = 0.5$ for $t \geq 0$, and the improper integral $\int_0^\infty f(t)\, dt$ certainly won't converge. If $c < 0$, then $f(t)$ is a positive increasing function, and again $\int_0^\infty f(t)\, dt$ won't converge. So, we must need to have $c > 0$.

   In this case,

$$\begin{aligned} 0.5 \int_0^\infty e^{-ct}\, dt &= 0.5 \lim_{b\to\infty} \int_0^b e^{-ct}\, dt \\ &= 0.5 \lim_{b\to\infty} -\frac{1}{c} e^{-ct}\Big|_0^b \\ &= 0.5 \lim_{b\to\infty} \left(-\frac{1}{c} e^{-cb} + \frac{1}{c}\right) \\ &= \frac{0.5}{c} \end{aligned}$$

   (Notice that, in the last step, we needed to use the fact that $c > 0$.) We want this to equal 1, so c should equal $\boxed{0.5}$.

   (b) *What is the probability that the toy lasts over one year? (Is there any way to compute this without computing an improper integral?)*

   Solution. The probability that the toy lasts over one year is given by the integral $\int_1^\infty f(t)\, dt$. If we want to avoid using an improper integral, we could instead calculate the probability that the toy lasts less than one year, which is $\int_0^1 f(t)\, dt$. Then, the probability that the toy lasts over one year is $1 - \int_0^1 f(t)\, dt$. Whichever method you use, the answer is $\boxed{\frac{1}{\sqrt{e}}}$.

   (c) *What is the median life of this type of toy?*

Solution. The median is the value T such that $\int_{-\infty}^{T} f(t)\,dt = \frac{1}{2}$. So, we want

$$\begin{aligned}\frac{1}{2} &= \int_0^T 0.5e^{-0.5t}\,dt \\ &= -e^{-0.5t}\Big|_0^T \\ &= -e^{-0.5T} + 1\end{aligned}$$

Solving, $T = \boxed{-2\ln\frac{1}{2}}$ (this is approximately 1.37 years).

2. *A large number of students take an exam. 30% of the students receive a score of 70, 50% receive a score of 80, and 20% receive a score of 90. What is the average score on the exam?*

   Solution. If the number of students is N, then $.3N$ people scored 70, $.5N$ scored 80, and $.2N$ scored 90. So, the sum of all scores is $.3N(70) + .5N(80) + .2N(90) = 79N$. The average score is this sum divided by the number of people, or $\boxed{79}$.

3. The density function for the duration of telephone calls within a certain city is $p(x) = 0.4e^{-0.4x}$ where x denotes the duration in minutes of a randomly selected call.

   (a) *What percentage of calls last one minute or less?*

   Solution. We are interested in the fraction of calls that last between 0 and 1 minute, which is $\int_0^1 0.4e^{-0.4x}\,dx = -e^{-0.4x}\Big|_0^1 = 1 - e^{-0.4}$. (As a percent, it is $100(1 - e^{-0.4}) \approx 33\%$.)

   (b) *What percentage of calls last between one and two minutes?*

   Solution. This is $\int_1^2 0.4e^{-0.4x}\,dx = -e^{-0.4x}\Big|_1^2 = e^{-0.4} - e^{-0.8}$, which is approximately 22%.

   (c) *What percentage of calls last 3 minutes or more?*

   Solution. This is

$$\begin{aligned}\int_3^\infty 0.4e^{-0.4x}\,dx &= \lim_{b\to\infty}\int_3^b 0.4e^{-0.4x}\,dx \\ &= \lim_{b\to\infty} -e^{-0.4x}\Big|_3^b \\ &= \lim_{b\to\infty} e^{-1.2} - e^{-0.4b} \\ &= e^{-1.2}\end{aligned}$$

   This works out to approximately 30%.

   (d) *What is the average length of a call?*

   Solution. The average length of a call is $\int_0^\infty x(0.4e^{-0.4x})\,dx$. Using integration by parts, an antiderivative of $x(0.4e^{-0.4x})$ is $-xe^{-0.4x} - 2.5e^{-0.4x}$. So,

$$\begin{aligned}\int_0^\infty x(0.4e^{-0.4x})\,dx &= \lim_{b\to\infty}\left(-xe^{-0.4x} - 2.5e^{-0.4x}\right)\Big|_0^b \\ &= \lim_{b\to\infty} -be^{-0.4b} - 2.5e^{-0.4b} + 2.5 \qquad (1)\end{aligned}$$

The middle term, $-2.5e^{-0.4b}$, tends to 0 as $b \to \infty$. For the first term, $-be^{-0.4b}$, we need to use L'Hospital's Rule (since the limit is of the form $\infty \cdot 0$):

$$\begin{aligned} \lim_{b\to\infty} -be^{-0.4b} &= \lim_{b\to\infty} -\frac{b}{e^{0.4b}} \\ &= \lim_{b\to\infty} -\frac{1}{0.4e^{0.4b}} \\ &= 0 \end{aligned}$$

Plugging this into (**??**), the average length of a call is $\boxed{\text{2.5 minutes}}$.

4. *The lifetime, in hundreds of hours, of a certain type of light bulb has been found empirically to have a probability density function approximated by* $f(x) = \frac{\sqrt{65}}{8(1+x^2)^{3/2}}$ *for* $0 < x < 8$. *Find the mean lifetime of such a bulb.*

Solution. The mean lifetime is $\displaystyle\int_0^8 xf(x)\,dx = \frac{\sqrt{65}}{8}\int_0^8 \frac{x}{(1+x^2)^{3/2}}\,dx$. To evaluate this, we substitute $u = 1 + x^2$:

$$\begin{aligned} \frac{\sqrt{65}}{8}\int_0^8 \frac{x}{(1+x^2)^{3/2}}\,dx &= \frac{\sqrt{65}}{8}\int_1^{65} \frac{1}{2}u^{-3/2}\,du \\ &= -\frac{\sqrt{65}}{8}u^{-1/2}\Big|_1^{65} \\ &= \frac{\sqrt{65}}{8}\left(1 - \frac{1}{\sqrt{65}}\right) \\ &= \boxed{\frac{\sqrt{65}-1}{8}} \end{aligned}$$