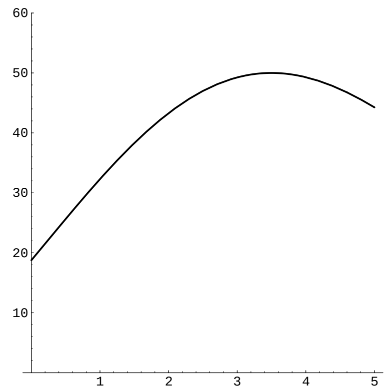
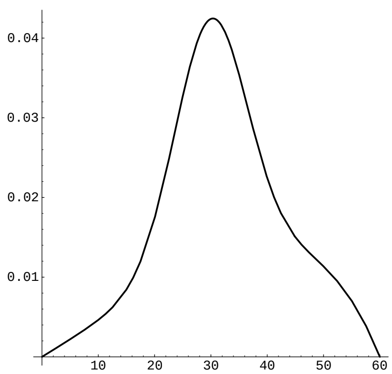
More Probability

1. (a) *The following function represents the temperature outside as a function of time. Estimate the average temperature between time 0 and time 5.*



Solution. We are looking for the average value of the temperature function, or the average height of its graph. In this graph, it looks like this is around 40. (The exact answer is around 41.5.)

(b) *A meteorologist takes several temperature readings which are described by the following probability density function. Estimate the average temperature.*



Solution. The probability density function has a peak at 30, but the area under the curve to the left is smaller than the area under the curve to the right. This tells us that more of the readings were above 30 than below 30, so we can guess that the average temperature was a bit above 30. It does not look like the average temperature will be as high as 40 because the fraction of temperature readings above 40 (the area under the graph to the right of 40) is pretty small. So, we guess that the average temperature is above 30 but below 40. (The actual value is about 32.)

2. Let $p(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$, a probability density function.

(a) *Sketch the graph of this probability density function. What do you think its mean is?*

Solution. It looks like 0 since the graph is symmetric about $x = 0$.

(b) *Verify your guess mathematically.*

Solution. To find the mean, we need to evaluate $\int_{-\infty}^{\infty} x \cdot \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\ dx$. Since the bounds $-\infty$ and ∞ are both improprieties, we need to split the integral into two pieces and evaluate:

$$\int_{-\infty}^{\infty} x \cdot \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\ dx = \frac{1}{\sqrt{2\pi}} \left(\int_{-\infty}^{0} xe^{-x^2/2}\ dx + \int_{0}^{\infty} xe^{-x^2/2}\ dx \right) \qquad (1)$$

Let's first find an antiderivative of $xe^{-x^2/2}$. We use the substitution $u = -x^2/2$ to get

$$\int xe^{-x^2/2}\ x = \int -e^u\ du = -e^u + C = -e^{-x^2/2} + C.$$

Now, we'll go back to (**??**). Let's do the integral from 0 to infinity first. Since it's an improper integral, we really need to take a limit:

$$\begin{aligned} \int_0^{\infty} xe^{-x^2/2}\ dx &= \lim_{b\to\infty} \int_0^b xe^{-x^2/2}\ dx \\ &= \lim_{b\to\infty} -e^{-x^2/2}\Big|_0^b \\ &= \lim_{b\to\infty} 1 - e^{-b^2/2} \\ &= 1 \end{aligned}$$

Similarly, we find that $\int_{-\infty}^{0} xe^{-x^2/2}\ dx = -1$, so (**??**) tells us that the mean is $-1 + 1 = \boxed{0}$.

3. *The bell curve with mean* 0 *and standard deviation* s *is given by the probability density function* $p(x) = \frac{1}{s\sqrt{2\pi}} e^{-x^2/(2s^2)}$. *What fraction of the population is within one standard deviation* s *of the mean 0?*

   Solution. We are looking for the fraction of the population that is between $-s$ and s, which is given by the integral $\boxed{\frac{1}{s\sqrt{2\pi}} \int_{-s}^{s} e^{-x^2/(2s^2)}\ dx}$.