Geometric Sums and Geometric Series

1. In your quest to become a millionaire by age 50, you start an aggressive savings plan. You open a new investment account on January 1, 2008 and deposit \$9000 into it every year on January 1. Each year, you earn 7% interest on December 31.

   (a) *How much money will you have in your account on January 2, 2009? 2010? 2014? (Don't try to add or multiply things out; just write an arithmetic expression.)*

   **Solution.** On December 31, 2008, you will have \$9000(1.07) because of the interest you've earned. After you deposit \$9000 on January 1, 2009, you will have $\$9000 + \$9000(1.07)$.

   On December 31, 2009, you will receive your interest, giving you $\$9000(1.07)+\$9000(1.07)^2$. After you make your yearly deposit on January 1, 2010, you will have $\$9000+\$9000(1.07)+\$9000(1.07)^2$.

   Continuing this reasoning, you will have $\$9000 + \$9000(1.07) + \$9000(1.07)^2 + \cdots + \$9000(1.07)^6$ on January 2, 2014.

   (b) *Will you be a millionaire by age 50?*

   **Solution.** The answer will depend on when you were born. Let's say that you were born in 1988. Then, we want to know what has happened by the year 2038. Using the argument of part (a), on January 2, 2038, you will have $\$9000 + \$9000(1.07) + \$9000(1.07)^2 + \cdots + \$9000(1.07)^{30}$. To see whether this is bigger than a million, we want some way to calculate this sum quickly.

   Let's call this amount $S$:

   $$S = 9000 + 9000(1.07) + 9000(1.07)^2 + 9000(1.07)^3 + \cdots + 9000(1.07)^{29} + 9000(1.07)^{30} \qquad (1)$$

   Notice that if we multiply both sides by 1.07, we get something similar looking on the right side:

   $$1.07S = 9000(1.07) + 9000(1.07)^2 + 9000(1.07)^3 + \cdots + 9000(1.07)^{30} + 9000(1.07)^{31} \qquad (2)$$

   Subtracting (**??**) from (**??**), most of the terms on the right side cancel, and we are left with $0.07S = 9000(1.07)^{31} - 9000$, so $S = \frac{9000(1.07)^{31}-9000}{.07} = \$918,657.37$. So, you are not quite a millionaire, but you are close!

2. If you suffer from allergies, your doctor may suggest that you take Claritin once a day. Each Claritin tablet contains 10 mg of loratadine (the active ingredient). Every 24 hours, about 7/8 of the loratadine in the body is eliminated (so 1/8 remains). [1]

   (a) *If you take one Claritin tablet every morning for a week, how much loratadine is in your body right after you take the 3rd tablet? 7th tablet? (Don't try to simplify your computations; just write out an arithmetic expression.)*

   **Solution.** Immediately after taking the first tablet, you have 10 mg of loratadine in your body. The following morning, only 1/8 of that is left, so you have 10 (1/8) mg in your body. You then take another pill containing 10 mg, so you have a total of $10 + 10(1/8)$ mg of loratadine in your body.

   The following morning, 1/8 of that remains, or $[10 + 10(1/8)](1/8) = 10(1/8) + 10(1/8)^2$. You then take another pill containing 10 mg, so you have a total of $10 + 10(1/8) + 10(1/8)^2$ mg of loratadine in your body after taking the 3rd pill.

[1] This estimate comes from the fact that the average half-life of loratadine is known to be 8 hours.

Continuing this reasoning, you will have $10+10(1/8)+10(1/8)^2+10(1/8)^3+10(1/8)^4+10(1/8)^5+10(1/8)^6$ mg in your body after the 7th pill.

(b) *If you take Claritin for years and years, will the amount of loratadine in your body level off? Or will your bloodstream be pure loratadine?*

**Solution.** Right after you take the $n$-th pill, you will have $10+10(1/8)+10(1/8)^2+10(1/8)^3+\cdots+10(1/8)^{n-1}$ mg of loratadine in your body. Let's call this amount $S_n$. We are wondering what happens to $S_n$ as $n$ gets very large.

We use the same trick we used in #1(b) to get a closed form expression for $S_n$. We said that

$$S_n = 10+10(1/8)+10(1/8)^2+\cdots+10(1/8)^{n-2}+10(1/8)^{n-1} \quad (3)$$

Multiplying both sides by 1/8, we get that

$$(1/8)S_n = 10(1/8)+10(1/8)^2+10(1/8)^3+\cdots+10(1/8)^{n-1}+10(1/8)^n \quad (4)$$

If we subtract (**??**) from (**??**), most of the terms on the right side cancel, and we are left with $(7/8)S_n = 10-10(1/8)^n$. Dividing both sides by 7/8, $S_n = \frac{10-10(1/8)^n}{7/8}$.

Using this expression for $S_n$, it is easy to see what happens as $n$ gets bigger and bigger: $(1/8)^n$ gets closer and closer to 0, so $\frac{10-10(1/8)^n}{7/8}$ gets closer and closer to $\frac{10}{7/8}=\frac{80}{7}$. Thus, over time, the amount of loratadine in your body gets closer and closer to $\frac{80}{7}$ mg.

3. *For what values of $x$ does the geometric series $1+x+x^2+\cdots$ converge?* [2] *If it converges, what does it converge to?*

**Solution.** This is a geometric series $a+ar+ar^2+ar^3+\cdots$ with $a=1$ and $r=x$. Therefore, we know that it diverges when $|x|\geq 1$. When $|x|<1$, it converges to $\frac{1}{1-x}$.

4. Which of the following series are geometric?

(a) $\displaystyle\sum_{k=1}^{\infty}\frac{(-1)^k 2^{k+1}}{3^k}$.

**Solution.** A geometric series is a series of the form $a+ar+ar^2+\cdots$. To decide whether the given series is geometric, we want to see if it matches this form. It's helpful to write out the first few terms. When $k=1$, we have $\frac{(-1)^1 2^2}{3^1}=-\frac{4}{3}$. When $k=2$, we have $\frac{(-1)^2 2^3}{3^2}=\frac{8}{9}$. When $k=3$, we have $\frac{(-1)^3 2^4}{3^3}=-\frac{16}{27}$. When $k=4$, we have $\frac{(-1)^4 2^5}{3^4}=\frac{32}{81}$. So far, it looks like it could be the geometric series $a+ar+ar^2+ar^3+\cdots$ with $a=-\frac{4}{3}$ and $r=-\frac{2}{3}$.

To see if this is correct, let's compare the $k$-th terms. The $k$-th term of the given series is $\frac{(-1)^k 2^{k+1}}{3^k}$, while the $k$-th term of the geometric series is $ar^{k-1}$. So, we are hoping that $\frac{(-1)^k 2^{k+1}}{3^k}=\left(-\frac{4}{3}\right)\left(-\frac{2}{3}\right)^{k-1}$. If you multiply out the right side, you will see that this is indeed the case, so the given series is geometric with $a=-\frac{4}{3}$ and $r=-\frac{2}{3}$.

(b) $\displaystyle\sum_{k=1}^{\infty}\frac{1}{k^3}$.

[2] We could also write this series in summation notation as $\displaystyle\sum_{k=0}^{\infty}x^k$.

**Solution.** If we write this series out, it is $1 + \frac{1}{8} + \frac{1}{27} + \frac{1}{64} + \cdots$. We can already see that it's not geometric because the terms don't have a common ratio. (To elaborate: if it was geometric, the first term would be $a$ and the second term would be $ar$; this means that $a$ would have to be 1, and $r$ would have to be $\frac{1}{8}$. But then the third term isn't right.)

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{2}{3^{n/2}}$.

**Solution.** We could also write this series as $\frac{2}{3^{1/2}} + \frac{2}{3} + \frac{2}{3^{3/2}} + \frac{2}{3^2} + \cdots$, which looks like it might be geometric with $a = \frac{2}{3^{1/2}}$ and $r = \frac{1}{3^{1/2}}$.

To check if this is correct, we want to see if the $n$-th term is $ar^{n-1} = \frac{1}{3^{1/2}} \left(\frac{1}{3^{1/2}}\right)^{n-1}$, and it is. So, the series is geometric with $a = \frac{2}{3^{1/2}}$ and $r = \frac{1}{3^{1/2}}$.