Series

1. Suppose you know that the infinite series $a_1 + a_2 + a_3 + \cdots + a_n + \cdots$ converges to s and that $a_k > 0$ for k any positive integer. Let $s_n = a_1 + a_2 + a_3 + \cdots + a_n$. For each of the following statements, determine whether the statement must be true, could possibly be true, or must be false.

   (a) $\lim_{n\to\infty} a_n = 0$.

   (b) $\lim_{n\to\infty} s_n = 0$.

   (c) There exists a number M such that $s_n < M$ for all n. (This is equivalent to saying that the partial sums are bounded. Why?)

   (d) $\sum_{k=5}^{\infty} a_k$ converges.

   Solution. (a) must be true, (b) must be false, (c) must be true, and (d) must be true. (See the solutions to Homework 15 for more details.)

2. *Suppose you know that* $\lim_{n\to\infty} b_n = 0$. *Can you be sure that the infinite series* $b_1+b_2+b_3+\cdots$ *converges?*

   Solution. No; the harmonic series in #5 is an example of a series that diverges even though its terms tend to 0.

3. (a) *Give an example of a sequence (ordered list) of numbers such that the numbers are increasing but are bounded.*

   Solution. $0.9, 0.99, 0.999, 0.9999, 0.99999, \ldots$ is one such example.

   (b) *Give an example of a sequence (ordered list) of numbers such that the numbers are increasing and are not bounded.*

   Solution. $1, 2, 3, 4, 5, \ldots$.

   (c) *Give an example of a sequence (ordered list) of numbers such that the numbers are bounded but have no limit as* $n \to \infty$.

   Solution. $0, 1, 0, 1, 0, 1, 0, 1, \ldots$.

4. (a) *A sequence which is both monotonic and bounded*

   must converge   *could either converge or diverge*   *must diverge*

   Solution. Must converge. This is the <u>Monotonic Sequence Theorem</u>.

   (b) *A sequence which is monotonic but not bounded*

   must converge   *could either converge or diverge*   *must diverge*

   Solution. Must diverge.

5. Consider the series $\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$ (called the <u>harmonic series</u>).

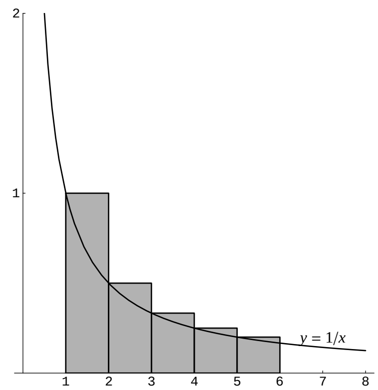
(a) *Does the sequence of terms converge? If so, to what does it converge?*

Solution. Yes, the terms $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$ are getting closer to 0.

(b) *Does the sequence of partial sums converge? If so, to what does it converge?*

Solution. The sequence of partial sums does not converge. The sequence of partial sums is definitely increasing (every partial sum is bigger than the previous partial sum). So, if we knew the sequence of partial sums was not bounded, we could conclude that it didn't converge (by #4(b)).

One way to see that the sequence of partial sums is not bounded is to look at them graphically. The partial sums can be represented as areas. For instance, the 5th partial sum is $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}$, which we can represent as the area in these 5 boxes:



As we can see, the 5th partial sum is therefore bigger than $\int_1^6 \frac{1}{x}\,dx$. In general, the n-th partial sum is bigger than $\int_1^{n+1} \frac{1}{x}\,dx$. Taking the limit as $n \to \infty$ of $\int_1^{n+1} \frac{1}{x}\,dx$ gives us the improper integral $\int_1^{\infty} \frac{1}{x}\,dx$, which we know diverges. So, the sequence of partial sums also diverges.

The other way to understand these partial sums is using a clever trick. Let's look at the partial sums $s_1, s_2, s_4, s_8, s_{16}, \ldots$.

$$
\begin{aligned}
s_1 &= 1 \\
s_2 &= 1 + \frac{1}{2} \\
s_4 &= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) \\
&> 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) \\
&= 1 + \frac{1}{2} + \frac{1}{2} \\
s_8 &= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) \\
&> 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) \\
&= 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2}
\end{aligned}
$$

$$\begin{aligned} s_{16} &= 1+\frac{1}{2}+\left(\frac{1}{3}+\frac{1}{4}\right)+\left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right)+\left(\frac{1}{9}+\frac{1}{10}+\cdots+\frac{1}{16}\right) \\ &> 1+\frac{1}{2}+\left(\frac{1}{4}+\frac{1}{4}\right)+\left(\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}\right)+\left(\frac{1}{16}+\frac{1}{16}+\cdots+\frac{1}{16}\right) \\ &= 1+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2} \end{aligned}$$

Since this pattern continues, there is no way that the sequence of partial sums can be bounded.

(c) *Does the series* $\sum_{k=1}^{\infty} \frac{1}{k}$ *converge?*

Solution. This is really the same question as (b); the sequence of partial sums diverges, so the series diverges.

(d) *Does the series* $\sum_{k=10^{10}}^{\infty} \frac{1}{100000000k}$ *converge?*

Solution. No. We know that $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges. Therefore, the series $\sum_{k=1}^{\infty} \frac{1}{100000000k}$ diverges (the whole thing has just been multiplied by a constant). We know that the beginning terms of a series do not affect its convergence, so if we start with the 10^{10}-th term, the series still diverges.

6. See if you can determine whether each of the following series converges or diverges by using the Nth Term Test for Divergence, results about geometric series, or some sort of comparison to series you know about.

(a) $\sum_{k=100}^{\infty} \frac{1}{3k}$.

Solution. This is basically the harmonic series multiplied by $\frac{1}{3}$, except that the first 99 terms are missing. Neither of things affects convergence, so this series $\boxed{\text{diverges}}$, just like the harmonic series.

(b) $\sum_{k=3}^{\infty} \frac{(-1)^k 2^k}{3^k}$.

Solution. We can rewrite this as $\sum_{k=3}^{\infty} \left(-\frac{2}{3}\right)^k$, which is a geometric series with common ratio $-\frac{2}{3}$. We know that such a series $\boxed{\text{converges}}$.

(c) $\sum_{k=4}^{\infty} \frac{(-1)^k 3^k}{2^k}$.

Solution. We can rewrite this as $\sum_{k=4}^{\infty} \left(-\frac{3}{2}\right)^k$, which is a geometric series with common ratio $-\frac{3}{2}$. We know that such a series $\boxed{\text{diverges}}$. Alternatively, you could use the Nth Term Test for Divergence: $\lim_{k\to\infty} \frac{(-1)^k 3^k}{2^k}$ does not exist, so the Nth Term Test for Divergence tells us that the series cannot converge either.

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n}$.

Solution. Intuitively, we know that a series converges if its terms go to 0 "quickly enough." The terms of this series do go to 0, but $\frac{\ln n}{n} > \frac{1}{n}$ when $n > e$, so the terms of the given series go to 0 more slowly than the terms of the harmonic series. Since we know that the harmonic series diverges, we should guess that this series diverges also.

To make it more precise mathematically, we could say that the partial sums of this series grow more quickly than the partial sums of the harmonic series. Since the partial sums of the harmonic series already grow without bound, the partial sums of this series also grow without bound, so this series $\boxed{\text{diverges}}$.

(e) $\displaystyle\sum_{n=2}^{\infty} \frac{n}{\ln n}$.

Solution. We can use the Nth Term Test for Divergence: $\displaystyle\lim_{n\to\infty} \frac{n}{\ln n} = \lim_{n\to\infty} \frac{1}{1/n} = \lim_{n\to\infty} n = \infty$ (using L'Hospital's Rule in the first step). Since the terms of this series do not tend to 0, the series $\boxed{\text{diverges}}$.

(f) $\displaystyle\sum_{n=0}^{\infty} \sin n$.

Solution. Again, we can use the Nth Term Test for Divergence: $\displaystyle\lim_{n\to\infty} \sin n$ does not exist, so the series $\boxed{\text{diverges}}$.