Comparison

1. Use the Comparison Test (also known as "direct comparison") to decide whether the following series converge or diverge.

(a) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}3^n}$.

Solution. When n is really big, 3^n is much much bigger than $\sqrt{n}$, so it seems like the terms of this series are affected more by 3^n than $\sqrt{n}$. Therefore, let's compare to the geometric series $\sum_{n=1}^{\infty} \frac{1}{3^n}$. When $n \geq 1$, $0 \leq \frac{1}{\sqrt{n}3^n} \leq \frac{1}{3^n}$. The geometric series $\sum_{n=1}^{\infty} \frac{1}{3^n}$ converges (the common ratio is $1/3$), so the Comparison Test tells us that $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}3^n}$ also converges.

(b) $\sum_{n=1}^{\infty} \frac{1}{n!}$.

Solution. If we "unpack" the summation notation, we get $1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} + \cdots$. These terms seem to go to 0 pretty quickly — certainly more quickly, say, than the geometric series $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$, which we know converges. So, let's compare to the geometric series $\sum_{n=1}^{\infty} \frac{1}{2^{n-1}}$.

(Why is it 2^{n-1} instead of 2^n, you might be wondering? Because we're starting with $n = 1$, but we want the first term in the geometric series to be 1.)

We'd like to say that $0 \leq \frac{1}{n!} \leq \frac{1}{2^{n-1}}$. The first inequality is obviously true. Now, $\frac{1}{n!}$ means $\frac{1}{n} \cdot \frac{1}{n-1} \cdot \frac{1}{n-2} \cdots \frac{1}{2}\frac{1}{1} = \frac{1}{n} \cdot \frac{1}{n-1} \cdot \frac{1}{n-2} \cdots \frac{1}{2}$, while $\frac{1}{2^{n-1}}$ means $\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdots \frac{1}{2}$. Both products have $n - 1$ terms, and all of the terms in the product for $\frac{1}{2^{n-1}}$ are at least as big as the corresponding term in $\frac{1}{n!}$. So, it is indeed the case that $0 \leq \frac{1}{n!} \leq \frac{1}{2^{n-1}}$. Since $\sum_{n=1}^{\infty} \frac{1}{2^{n-1}}$ converges (it's a geometric series with common ratio $1/2$), the Comparison Test tells us that $\sum_{n=1}^{\infty} \frac{1}{n!}$ also converges.

(c) $\sum_{n=1}^{\infty} \frac{n^2}{n^3 - n + 1000}$.

Solution. You might have the intuition that the terms $\frac{n^2}{n^3-n+1000}$ "grow like" $\frac{n^2}{n^3} = \frac{1}{n}$, so the series should act like $\sum_{n=1}^{\infty} \frac{1}{n}$ and diverge. To verify this, we should compare to $\sum_{n=1}^{\infty} \frac{1}{n}$.

Let's try: we want to say that $0 \leq \frac{1}{n} \leq \frac{n^2}{n^3-n+1000}$. The first inequality is always true, but the second inequality is only true when $n^3 \geq n^3 - n + 1000$, or $n \geq 1000$. This isn't a big problem though; as we know, the beginning terms of a series don't affect convergence.

So, here's the appropriate reasoning: $0 \leq \frac{1}{n} \leq \frac{n^2}{n^3-n+1000}$ when $n \geq 1000$. We know that $\sum_{n=1000}^{\infty} \frac{1}{n}$ diverges (this is the harmonic series without the first 999 terms, and we know the harmonic

series diverges). Therefore, the Comparison Test tells us that $\displaystyle\sum_{n=1000}^{\infty} \frac{n^2}{n^3 - n + 1000}$ also diverges.

Adding in finitely many terms at the beginning doesn't affect convergence, so $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{n^3 - n + 1000}$ also diverges.

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\ln(1+n)}$.

Solution. The only series we understand so far are geometric series and the harmonic series. We know that $\ln x$ grows more slowly than x, so it seems like we should compare this to the harmonic series.

By graphing $\ln(1+x)$ and x, we can see that $\ln(1+x) \leq x$ when $x \geq 0$, so $\frac{1}{\ln(1+n)} \geq \frac{1}{n} \geq 0$ for $n \geq 1$. Therefore, the Comparison Test tells us that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\ln(1+n)}$ diverges.

(To prove that $\ln(1+x) \leq x$, you could do something like this: using what you learned in Math 1a, you can show that the global minimum of $x - \ln(1+x)$ is 0. This means that $x - \ln(1+x) \geq 0$ for all x, so $x \geq \ln(1+x)$.)

2. *True or false: If $\{a_n\}$ is a sequence with positive terms and $\displaystyle\lim_{n\to\infty} a_n = 0$, then there is a number k such that $a_n < 1$ whenever $n \geq k$.*

Solution. Roughly, the statement is saying that, if $\{a_n\}$ is a sequence of positive numbers whose limit is 0, then after a while, all of the numbers in the sequence must be less than 1. This is true; after all, if the limit is 0, that means all of the terms in the sequence must stay really close to 0 after a while.

3. Decide whether the following series converge or diverge using any method you like.

(a) $\displaystyle\sum_{n=100}^{\infty} \cos n$.

Solution. We know that a series converges if its terms go to 0 "quickly enough". In this case, the terms aren't going to 0 at all!

To make this precise, $\displaystyle\lim_{n\to\infty} \cos n$ does not exist, so the Nth Term Test for Divergence says that the series will diverge.

(b) $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k 2^{k+1}}{3^k}$.

Solution. We can rewrite the k-th term as $\frac{(-1)^k \cdot 2 \cdot 2^k}{3^k} = 2\left(\frac{-2}{3}\right)^k$, so the series is $\displaystyle\sum_{k=1}^{\infty} 2\left(\frac{-2}{3}\right)^k$.

This is a geometric series whose first term is $-\frac{4}{3}$ and whose common ratio is $-\frac{2}{3}$. Therefore, we know that this series converges.

(c) $1 + 0 + (-1) + 1 + 0 + (-1) + 1 + 0 + (-1) + \cdots$.

Solution. One way to approach this is simply to write down the sequence of partial sums: $1, 1, 0, 1, 1, 0, 1, 1, 0, \ldots$. Since the sequence of partial sums diverges, the series diverges. (This is using the *definition* of convergence/divergence of a series, not any particular test.)

2

Alternatively, we could use the Nth Term Test for Divergence: the sequence of terms is $a_1 = 1, a_2 = 0, a_3 = -1, a_4 = 1, a_5 = 0, a_6 = -1, \ldots$. Since $\lim_{n \to \infty} a_n$ does not exist, the Nth Term Test for Divergence says that the series must diverge.

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n}$.

Solution. We can compare this to the harmonic series: $\frac{\ln n}{n} \geq \frac{1}{n} \geq 0$ as long as $n \geq 3$. The series $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n}$ diverges, so the Comparison Test tells us that $\displaystyle\sum_{n=3}^{\infty} \frac{\ln n}{n}$ diverges as well. Adding on finitely many terms at the beginning doesn't change whether a series converges or diverges, so $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n}$ diverges.