More Comparison

1. *For what values of p is the* <u>*p-series*</u> $\sum_{n=1}^{\infty} \frac{1}{n^p}$ *convergent?*

   Solution. If we let $f(x) = \frac{1}{x^p}$, then the terms of our series are just $f(1), f(2), f(3), \ldots$. Also, $f(x)$ is positive, continuous, and decreasing on $[1, \infty)$, so we can use the Integral Test. The Integral Test says $\sum_{n=1}^{\infty} \frac{1}{n^p}$ and $\int_1^{\infty} \frac{1}{x^p}\, dx$ either both converge or both diverge.

   We know that $\int_1^{\infty} \frac{1}{x^p}$ converges when $p > 1$ and diverges when $p \le 1$. So, the p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges when $p > 1$ and diverges when $p \le 1$.

2. *Does the series* $\sum_{n=2}^{\infty} \frac{1}{n^2 - 1}$ *converge or diverge?*

   Solution. This looks quite similar to $\sum_{n=2}^{\infty} \frac{1}{n^2}$, but direct comparison doesn't work too well because $\frac{1}{n^2-1} \ge \frac{1}{n^2}$. Instead, we'll use the Limit Comparison Test and compare to $\sum_{n=2}^{\infty} \frac{1}{n^2}$.

   We have $\lim_{n\to\infty} \frac{1/(n^2-1)}{1/n^2} = \lim_{n\to\infty} \frac{n^2}{n^2-1} = 1$. Since $\sum_{n=2}^{\infty} \frac{1}{n^2}$ converges (by #1), the Limit Comparison Test says that $\sum_{n=2}^{\infty} \frac{1}{n^2-1}$ converges.

3. *Does the series* $\sum_{k=1}^{\infty} \frac{3}{8^k - 2}$ *converge or diverge?*

   Solution. When k is large, $\frac{3}{8^k-2} \approx \frac{3}{8^k}$. The series $\sum_{k=1}^{\infty} \frac{3}{8^k}$ is geometric with common ratio $\frac{1}{8}$, so it converges. Therefore, we expect our series to converge as well. However, we can't compare directly because $\frac{3}{8^k-2} \ge \frac{3}{8^k}$. Instead, we'll use the Limit Comparison Test to compare these two series.

   We have $\lim_{k\to\infty} \frac{3/(8^k-2)}{3/8^k} = \lim_{k\to\infty} \frac{8^k}{8^k-2} = 1$. Since $\sum_{k=1}^{\infty} \frac{3}{8^k}$ converges, the Limit Comparison Test says that $\sum_{k=1}^{\infty} \frac{3}{8^k-2}$ converges as well.

4. *Otto is given the following problem for homework.*

   > *Decide whether the series* $\sum_{n=1}^{\infty} \sin^2(\pi n)$ *converges or diverges. Explain your reasoning.*

   Otto writes

The improper integral $\int_1^\infty \sin^2(\pi x)\,dx$ *diverges, so* $\sum_{n=1}^{\infty} \sin^2(\pi n)$ *also diverges by the Integral Test.*

Otto is correct that the improper integral diverges (although he should have shown more work!). But the rest of his reasoning is incorrect — why? And what is the correct answer to the problem?

Solution. The Integral Test doesn't apply to Otto's problem because the function $\sin^2(\pi x)$ is not a decreasing function. In fact, using the Integral Test gives Otto the wrong answer: $\sin^2(\pi n) = 0$ whenever n is an integer, so the series in Otto's problem is just $0 + 0 + 0 + \cdots$, which converges.

5. Let $\sum a_n$ and $\sum b_n$ be series with positive terms. The Limit Comparison Test only applies when $\lim_{n\to\infty} \frac{a_n}{b_n}$ is a positive real number.

   (a) *If* $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$*, can you draw any conclusions?*

   Solution. Intuitively, the fact that $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ means that a_n goes to 0 a lot more quickly than b_n. So, if $\sum b_n$ converges, then $\sum a_n$ converges.

   In fact, you showed this in Problem Set 17 (#38a from Stewart §8.3).

   If $\sum b_n$ diverges, $\sum a_n$ could still converge; an example is $a_n = \frac{1}{n}$ and $b_n = 1$.

   (b) *If* $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$*, can you draw any conclusions?*

   Solution. Intuitively, the fact that $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$ means that a_n goes to 0 a lot more slowly than b_n. So, if $\sum b_n$ diverges, then $\sum a_n$ diverges as well.

   To make this mathematically correct, we can use the Comparison Test. The fact that $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$ means that, eventually (when n is big enough), $\frac{a_n}{b_n}$ will always be greater than 1. So, eventually, $a_n > b_n$, and the Comparison Test says that $\sum a_n$ will have to diverge since $\sum b_n$ does. (Just like in #1(c) from the "Comparison" handout, it's not true that $a_n > b_n$ for all n, just when n is big enough. But that's fine because we know the beginning terms of a series don't affect whether it converges.)

   If $\sum b_n$ converges, we can't draw any conclusion about $\sum a_n$.