Absolute and Conditional Convergence

1. Does the series $\sum_{k=1}^{\infty}(-1)^{k+1}\frac{1}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots$ converge or diverge? (This series is often called the alternating harmonic series.)

2. In fact, $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots = \ln 2$. Write a finite sum which estimates $\ln 2$ with error of less than 0.001. Is your approximation too big or too small?

Definition. *An alternating series is a series whose terms are alternately positive and negative. An alternating series can be written as either*

$$\begin{aligned} a_1 - a_2 + a_3 - \cdots &= \sum_{k=1}^{\infty}(-1)^{k+1}a_k \\ \text{or} \quad -a_1 + a_2 - a_3 + \cdots &= \sum_{k=1}^{\infty}(-1)^{k}a_k \end{aligned}$$

with all $a_k > 0$. (Which one depends on whether the first term is positive or negative.)

Definition. *A series $\sum a_n$ is called absolutely convergent if the series of absolute values $\sum |a_n|$ is convergent.*

3. Is the alternating harmonic series $\sum_{k=1}^{\infty}(-1)^{k+1}\frac{1}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots$ absolutely convergent?

Theorem. *If the series $\sum a_n$ is absolutely convergent, then it is convergent.*

The opposite is not true: there are convergent series which are not absolutely convergent. These series are called conditionally convergent.

4. Determine whether each series converges or diverges. If it converges, does it converge absolutely or conditionally?

   (a) $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k}{\sqrt{k}}$.

   (b) $\displaystyle\sum_{k=1}^{\infty} \frac{\sin k}{k!}$.

   (c) $\displaystyle\sum_{k=1}^{\infty} (-2)^k$.

5. (a) The Taylor series for $\cos x$ about 0 is $1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \displaystyle\sum_{k=0}^{\infty} (-1)^{k+1} \frac{x^{2k}}{(2k)!}$. Show that, if you plug in any value of x with $-0.5 \le x \le 0.5$, the series converges.

   (b) In fact, the series converges for all x, and $\cos x$ is actually equal to the series; that is,

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots .$$

   Suppose you use the approximation $\cos x \approx 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$ to approximate $\cos x$ when $-0.5 \le x \le 0.5$. Find an upper bound for the error. (This means: find a number U that you can show is bigger than the error.)