Absolute and Conditional Convergence

1. *Does the series* $\displaystyle\sum_{k=1}^{\infty}(-1)^{k+1}\frac{1}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots$ *converge or diverge? (This series is often called the alternating harmonic series.)*

   Solution. To see if the series converges, let's see if we can apply the Alternating Series Test. The terms are decreasing in magnitude: $\frac{1}{k+1} \leq \frac{1}{k}$. In addition, the terms approach 0: $\displaystyle\lim_{k\to\infty}\frac{1}{k} = 0$. Therefore, the Alternating Series Test applies, and we can conclude that $\displaystyle\sum_{k=1}^{\infty}(-1)^{k+1}\frac{1}{k}$ converges.

2. *In fact,* $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots = \ln 2$. *Write a finite sum which estimates* $\ln 2$ *with error of less than* 0.001. *Is your approximation too big or too small?*

   Solution. The Alternating Series Estimate tells us that the magnitude of the error in using the n-th partial sum $s_n = \displaystyle\sum_{k=1}^{n}(-1)^{k+1}\frac{1}{k}$ is at most the magnitude of the next term, or $\frac{1}{n+1}$. That is, $|s - s_n| \leq \frac{1}{n+1}$. We want $|s - s_n| < 0.001$, so let's find an n satisfying $\frac{1}{n+1} < 0.001$. (Then we'll have $|s - s_n| \leq \frac{1}{n+1} < 0.001$.)

   To get $\frac{1}{n+1} < 0.001$, we need $n + 1 > 1000$. The smallest n which makes this work is $n = 1000$, so we can use the 1000th partial sum $\boxed{1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots - \frac{1}{999} + \frac{1}{1000}}$.

   To determine whether our approximation is too big or too small, let's go back and look at our diagram of partial sums. Here it is, with the actual sum s in red:
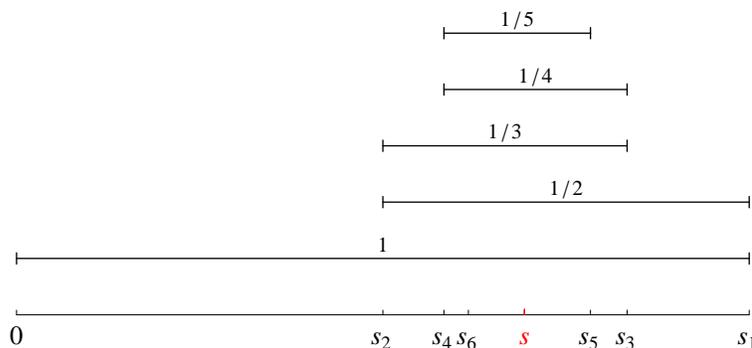
   From the diagram, we can see that the even partial sums $s_2, s_4, s_6, \ldots$ are all $\boxed{\text{too small}}$.

3. *Is the alternating harmonic series* $\displaystyle\sum_{k=1}^{\infty}(-1)^{k+1}\frac{1}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots$ *absolutely convergent?*

1

Solution. Asking whether the series is absolutely convergent is the same as asking whether the series $\sum_{n=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots$ converges. This is the harmonic series, which diverges. Thus, the series $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k}$ is not absolutely convergent.

4. Determine whether each series converges or diverges. If it converges, does it converge absolutely or conditionally?

(a) $\sum_{k=1}^{\infty} \frac{(-1)^k}{\sqrt{k}}$.

Solution. We can apply the Alternating Series Test: $\frac{1}{\sqrt{k+1}} \leq \frac{1}{\sqrt{k}}$ and $\lim_{k \to \infty} \frac{1}{\sqrt{k}} = 0$, so the Alternating Series Test says that $\sum_{k=1}^{\infty} \frac{(-1)^k}{\sqrt{k}}$ converges.

To decide whether it converges absolutely, we look at the series of absolute values, $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$. This is a p-series with $p = \frac{1}{2}$, and we know that diverges. So, $\sum_{k=1}^{\infty} \frac{(-1)^k}{\sqrt{k}}$ $\boxed{\text{converges conditionally}}$.

(b) $\sum_{k=1}^{\infty} \frac{\sin k}{k!}$.

Solution. This series is not an alternating series, so we should not even try the Alternating Series Test. All of our other tests involve series with positive terms, so let's look at the series of absolute values first. That is, we'll first look at $\sum_{k=1}^{\infty} \frac{|\sin k|}{k!}$.

We can use the Comparison Test here: $0 \leq \frac{|\sin k|}{k!} \leq \frac{1}{k!}$. We know that $\sum_{k=1}^{\infty} \frac{1}{k!}$ converges (see #1(b) from the "Comparison" handout), so $\sum_{k=1}^{\infty} \frac{|\sin k|}{k!}$ converges.

This means that $\sum_{k=1}^{\infty} \frac{\sin k}{k!}$ $\boxed{\text{converges absolutely}}$. We know that a series which converges absolutely also converges.

(c) $\sum_{k=1}^{\infty} (-2)^k$.

Solution. This is an alternating series, but the Alternating Series Test does not apply because the terms are not decreasing in magnitude. That does tell us whether the series converges though.

Instead, notice that the terms aren't going to 0: $\lim_{k \to \infty} (-2)^k$ does not exist. So, the Nth Term Test for Divergence says that the series $\boxed{\text{diverges}}$. (Alternatively, you could justify by saying that the series is geometric with common ratio -2.)

2

5. (a) *The Taylor series for $\cos x$ about 0 is $1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \dfrac{x^6}{6!} + \cdots = \sum_{k=0}^{\infty} (-1)^{k+1} \dfrac{x^{2k}}{(2k)!}$. Show that, if you plug in any value of x with $-0.5 \le x \le 0.5$, the series converges.*

Solution. We will show that we can apply the Alternating Series Test. If $-0.5 \le x \le 0.5$, then $\dfrac{x^{2(k+1)}}{[2(k+1)]!} \le \dfrac{x^{2k}}{(2k)!}$. This is true because the expression on the left has a bigger denominator and smaller numerator than the expression on the right (in the numerator, x is raised to a higher exponent, and $|x| < 1$). So, the first condition in the Alternating Series Test is satisfied.

Next, we need to show that $\lim\limits_{k \to \infty} \dfrac{x^{2k}}{(2k)!} = 0$. In fact, $\lim\limits_{k \to \infty} x^{2k} = 0$ (since x is between -0.5 and 0.5), so $\lim\limits_{k \to \infty} \dfrac{x^{2k}}{(2k)!} = 0$ as well.

Therefore, the Alternating Series Test applies, and we can conclude that the series converges.

(b) *In fact, the series converges for all x, and $\cos x$ is actually equal to the series; that is,*

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots .$$

Suppose you use the approximation $\cos x \approx 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$ to approximate $\cos x$ when $-0.5 \le x \le 0.5$. Find an upper bound for the error. (This means: find a number U that you can show is bigger than the error.)

Solution. The Alternating Series Estimation Theorem tells us that the magnitude of the error is at most that of the first unused term, which is $\frac{x^6}{6!}$. Since $|x| \le 0.5$, the error is at most $\frac{0.5^6}{6!} \approx 0.0000217014$.