

Ratio Test

1. What does the Ratio Test tell you about the following series?

(a) $\sum_{k=0}^{\infty} (-1)^{k+1} \frac{1000^k}{k!}$.

Solution. Since $\lim_{k \rightarrow \infty} \left| \frac{(-1)^{(k+1)+1} \frac{1000^{k+1}}{(k+1)!}}{(-1)^{k+1} \frac{1000^k}{k!}} \right| = \lim_{k \rightarrow \infty} \frac{1000^{k+1} \cdot k!}{(k+1)! \cdot 1000^k} = \lim_{k \rightarrow \infty} \frac{1000}{k+1} = 0$, the Ratio Test says that the series converges absolutely.

(b) $\sum_{k=1}^{\infty} \frac{1}{k}$.

Solution. $\lim_{k \rightarrow \infty} \left| \frac{\frac{1}{k+1}}{\frac{1}{k}} \right| = \lim_{k \rightarrow \infty} \left| \frac{k}{k+1} \right| = 1$, so the Ratio Test is inconclusive. (Of course, we know the series diverges.)

(c) $\sum_{k=1}^{\infty} \frac{1}{k^2}$.

Solution. $\lim_{k \rightarrow \infty} \left| \frac{\frac{1}{(k+1)^2}}{\frac{1}{k^2}} \right| = \lim_{k \rightarrow \infty} \left| \frac{k^2}{(k+1)^2} \right| = 1$, so the Ratio Test is inconclusive. (Of course, we know the series converges.)

2. When we studied Taylor series, we found that the Taylor series for $\sin x$ about 0 was $x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$, which can be written in summation notation as $\sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}$. For what values of x does this series converge?

Solution. We'll use the Ratio Test:

$$\begin{aligned} \lim_{k \rightarrow \infty} \left| \frac{(-1)^{k+1} \frac{x^{2(k+1)+1}}{[2(k+1)+1]!}}{(-1)^k \frac{x^{2k+1}}{(2k+1)!}} \right| &= \lim_{k \rightarrow \infty} \left| \frac{x^{2k+3} \cdot (2k+1)!}{(2k+3)! \cdot x^{2k+1}} \right| \\ &= \lim_{k \rightarrow \infty} \left| \frac{x^2}{(2k+3)(2k+2)} \right| \\ &= 0. \end{aligned}$$

(Since we're taking the limit as k tends to infinity, we treat x as a constant when taking the limit.) Therefore, the Ratio Test says that, no matter what x is, the series converges absolutely for all x .

3. When we studied Taylor series, we found that the Taylor series for $\ln(1+x)$ about 0 was $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$, which can be written in summation notation as $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{x^k}{k}$. For what values of x does this series converge?

Solution. We use the Ratio Test:

$$\begin{aligned} \lim_{k \rightarrow \infty} \left| \frac{(-1)^{(k+1)+1} \frac{x^{k+1}}{k+1}}{(-1)^{k+1} \frac{x^k}{k}} \right| &= \lim_{k \rightarrow \infty} \left| \frac{x^{k+1} \cdot k}{(k+1) \cdot x^k} \right| \\ &= \lim_{k \rightarrow \infty} \left| x \frac{k}{k+1} \right| \\ &= |x| \end{aligned}$$

By the Ratio Test, the series converges absolutely when $|x| < 1$ and diverges when $|x| > 1$. However, the Ratio Test is inconclusive when $|x| = 1$, so we'll have to test $x = \pm 1$ separately.

When $x = 1$, the series is $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$. This is the alternating harmonic series, and we've seen that this converges conditionally. (See #1 from the "Absolute and Conditional Convergence" handout.)

When $x = -1$, the series is $-1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{4} - \dots$, which is just -1 times the harmonic series, and we know that this diverges.

So, our final answer is:

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| $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{x^k}{k} \begin{cases} \text{converges absolutely} & \text{when } x < 1 \\ \text{diverges} & \text{when } x > 1 \text{ or } x = -1 \\ \text{converges conditionally} & \text{when } x = 1 \end{cases}$ |
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4. Decide whether the following series converge absolutely, converge conditionally, or diverge. You may use any method you like, but explain your reasoning. There is one that you will not be able to do (this is not due to a personal failing; it's just that all of the tests that we know are inconclusive).

(a) $\sum_{n=1}^{\infty} \frac{\cos n}{n^2}$.

Solution. When sin or cos appears in a series, it's often helpful to use the comparison test and the fact that $|\cos x|, |\sin x| \leq 1$. Remember that we need a series with positive terms to use the comparison test. So, let's look at the series of absolute values, $\sum_{n=1}^{\infty} \frac{|\cos n|}{n^2}$.

Since $0 \leq \frac{|\cos n|}{n^2} \leq \frac{1}{n^2}$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges (it's a p -series with $p = 2$), the Comparison Test says

that $\sum_{n=1}^{\infty} \frac{|\cos n|}{n^2}$ converges.

This tells us that $\sum_{n=1}^{\infty} \frac{\cos n}{n^2}$ converges absolutely.

(b) $\sum_{n=100}^{\infty} \frac{n!n!}{(2n)!}$.

Solution. This has lots of factorials, so the Ratio Test is a good test to try.

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{\frac{(n+1)!(n+1)!}{(2n+2)!}}{\frac{n!n!}{(2n)!}} \right| &= \lim_{n \rightarrow \infty} \frac{(n+1)(n+1)(2n)!}{n!n!(2n+2)!} \\ &= \lim_{n \rightarrow \infty} \frac{(n+1)(n+1)}{(2n+1)(2n+2)} \\ &= \lim_{n \rightarrow \infty} \frac{n^2 + 2n + 1}{4n^2 + 6n + 2} \\ &= \frac{1}{4} \end{aligned}$$

Therefore, the Ratio Test says that the series converges absolutely.

(c) $\sum_{n=0}^{\infty} \frac{\sin n}{n}$.

Solution. None of the tests we know work here.

(d) $\sum_{n=2}^{\infty} \frac{\ln n}{n}$.

Solution. We will use the Comparison Test: $0 \leq \frac{1}{n} \leq \frac{\ln n}{n}$ when $n > e$. We know that $\sum_{n=3}^{\infty} \frac{1}{n}$ diverges (it's the harmonic series), so the Comparison Test says that $\sum_{n=3}^{\infty} \frac{\ln n}{n}$ diverges, too. Adding on a term at the beginning doesn't affect convergence, so $\sum_{n=2}^{\infty} \frac{\ln n}{n}$ also diverges.

(e) $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{n^3 + 1}$.

Solution. When n is really big, $(-1)^{n+1} \frac{n}{n^3+1} \approx (-1)^{n+1} \frac{n}{n^3} = (-1)^{n+1} \frac{1}{n^2}$. We know that the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges (it's a p -series with $p = 2$), so we can guess that the given series is probably absolutely convergent.

To verify this, we'll compare the series of absolute values $\sum_{n=1}^{\infty} \frac{n}{n^3 + 1}$ to $\sum_{n=1}^{\infty} \frac{1}{n^2}$ using the Limit Comparison Test (we are allowed to use the Limit Comparison Test because both of these series have positive terms; however, we may not use the Limit Comparison Test with the original series since it has both positive and negative terms).

Since $\lim_{n \rightarrow \infty} \frac{\frac{n}{n^3+1}}{\frac{1}{n^2}} = \lim_{n \rightarrow \infty} \frac{n^3}{n^3+1} = 1$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, $\sum_{n=1}^{\infty} \frac{n}{n^3+1}$ converges by the Limit Comparison Test. This means that $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{n^3+1}$ converges absolutely.

$$(f) \sum_{n=5000}^{\infty} (-1)^n \frac{n}{n+1}.$$

Solution. When n is really big, $(-1)^{n+1} \frac{n}{n+1} \approx (-1)^{n+1} \frac{n}{n} = (-1)^{n+1}$. These aren't going to 0, so we should use the Nth Term Test for Divergence: $\lim_{n \rightarrow \infty} (-1)^n \frac{n}{n+1}$ does not exist (half of the terms are getting closer to 1 while the other half are getting closer to -1), so the given series diverges.