Power Series

A <u>power series</u> centered at the number a is a series of the form $\sum_{n=0}^{\infty} c_n(x-a)^n$ where x is a variable and the c_n are constants.

1. For what values of x does the power series $\sum_{n=1}^{\infty} n!x^n$ converge? (This series is centered at 0.)

Theorem. For a given power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ centered at a, there are 3 possibilities:

1. The series converges only when $x = a$.
2. The series converges for all x.
3. There is a positive number R such that the series converges when $|x-a| < R$ and diverges when $|x-a| > R$. R is called the <u>radius of convergence</u>. (Note that this doesn't say anything about what happens when $|x-a| = R$.)

The <u>interval of convergence</u> of a power series is the set of x for which the power series converges.

2. Find the radius of convergence and interval of convergence of the power series $\sum_{n=1}^{\infty} \frac{2^n}{n}(x-3)^n$.

3. We know that the power series $\sum_{n=0}^{\infty} x^n$ converges to $\frac{1}{1-x}$ when $|x| < 1$. Find a power series representation of the function $\frac{x}{1+4x^2}$. What is the radius of convergence of this power series?

Theorem. If the power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ has radius of convergence R where $R > 0$ or $R = \infty$, then the function $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$ is differentiable on the interval $(a-R, a+R)$ and

1. $f'(x) = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1}$.
2. $\int f(x)\, dx = C + \sum_{n=0}^{\infty} \frac{c_n}{n+1}(x-a)^{n+1}$.

The power series in (1) and (2) both have radius of convergence R. (Note: Although the radius of convergence remains unchanged, the interval of convergence may change.)

4. (a) Find a power series representation for $\ln(1+x)$ centered at 0. What is the radius of convergence for the power series you have found? (Hint: What is the derivative of $\ln(1+x)$?)

   (b) Find the degree 5 Taylor polynomial approximation of $\ln(1+x)$.

5. Find a power series representation of $\arctan(5x)$ centered at 0. What is the radius of convergence of the power series you have found?