More on Power Series

1. Suppose we have a power series $\sum_{n=1}^{\infty} c_n(x+7)^n$.

   (a) If you know that the power series converges when $x = 0$, what conclusions can you draw?

   (b) Suppose you also know that the power series diverges when $x = 1$. Now what conclusions can you draw?

   (c) Does $\sum_{n=1}^{\infty} c_n$ converge (assuming that the power series converges when $x = 0$ and diverges when $x = 1$)?

   (d) Does $\sum_{n=1}^{\infty} c_n(-8.1)^n$ converge?

   (e) Does $\sum_{n=1}^{\infty} c_n(-8)^n$ converge?

Theorem. If $f(x)$ has a power series representation centered at a (that is, $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$ for $|x-a| < R$), then that power series must be the Taylor series of f at a.

2. (a) Taking for granted that $\sin x = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!}$ for all x, find the Taylor series of $x\sin(x^3)$ at 0.

   (b) What is the radius of convergence of the power series you found in part (a)?

   (c) Let $f(x) = x\sin(x^3)$. What is $f'''(0)$? $f^{(4)}(0)$?

Theorem. If the power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ has radius of convergence R where $R > 0$ or $R = \infty$, then the function $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$ is differentiable on the interval $(a-R, a+R)$ and

(a) $f'(x) = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1}$.

(b) $\int f(x)\,dx = C + \sum_{n=0}^{\infty} \frac{c_n}{n+1}(x-a)^{n+1}$.

The power series in (1) and (2) both have radius of convergence R. (Note: Although the radius of convergence remains unchanged, the interval of convergence may change.)

3. (a) Find a power series representation of $\arctan(5x)$ centered at 0.

(b) What is the radius of convergence of the power series you found in part (a)?

4. In each part, find a power series that has the given interval of convergence. (Hint: If you get stuck, try finding the interval of convergence of $\sum_{n=1}^{\infty} \frac{x^n}{n}$.)

(a) $(-6, 0)$.

(b) $(-1, 3)$.

(c) Challenge: $[-1, 3)$.