

More on Power Series

1. Suppose we have a power series $\sum_{n=1}^{\infty} c_n(x+7)^n$.

(a) *If you know that the power series converges when $x = 0$, what conclusions can you draw?*

Solution. The power series is centered at -7 , so the fact that it converges at $x = 0$ means that the interval of convergence is at least $(-14, 0]$.

(b) *Suppose you also know that the power series diverges when $x = 1$. Now what conclusions can you draw?*

Solution. The interval of convergence is at most $[-15, 1)$.

(c) *Does $\sum_{n=1}^{\infty} c_n$ converge (assuming that the power series converges when $x = 0$ and diverges when $x = 1$)?*

Solution. This is the power series when $x+7 = 1$, or $x = -6$. In part (a), we found that -6 must be in the interval of convergence, so the series converges.

(d) *Does $\sum_{n=1}^{\infty} c_n(-8.1)^n$ converge?*

Solution. This is the power series when $x+7 = -8.1$, or $x = -15.1$. In part (b), we found that -15.1 cannot be in the interval of convergence, so the series diverges.

(e) *Does $\sum_{n=1}^{\infty} c_n(-8)^n$ converge?*

Solution. This is the power series when $x+7 = -8$, or $x = -15$. Neither (a) nor (b) tells us what must happen, so there is not enough information to determine whether the series converges.

2. (a) *Taking for granted that $\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$ for all x , find the Taylor series of $x \sin(x^3)$ at 0 .*

Solution. The theorem tells us that, if we can find a power series representation of $x \sin(x^3)$, then that is the Taylor series. So, rather than trying to find the Taylor series directly (by taking derivatives), let's look for a power series representation of $x \sin(x^3)$.

We are told that

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \text{ for all } x.$$

Replacing x by x^3 everywhere gives:

$$\sin x^3 = \sum_{n=0}^{\infty} (-1)^n \frac{x^{6n+3}}{(2n+1)!},$$

still true for all x since the first equation was true for all x . Now, we multiply both sides by x to get

$$x \sin x^3 = \sum_{n=0}^{\infty} (-1)^n \frac{x^{6n+4}}{(2n+1)!},$$

still for all x .

- (b) *What is the radius of convergence of the power series you found in part (a)?*

Solution. We said in part (a) that the power series representation was valid for all x , so the radius of convergence of the series must be $\boxed{\infty}$.

- (c) *Let $f(x) = x \sin(x^3)$. What is $f'''(0)$? $f^{(4)}(0)$?*

Solution. In part (a), we found that $f(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{6n+4}}{(2n+1)!}$. Unpacking the summation notation, $f(x) = x^4 - \frac{x^{10}}{3!} + \frac{x^{16}}{5!} - \dots$. Using this fact, there are two ways to solve the problem.

The slick way: The theorem says that, since $f(x) = x^4 - \frac{x^{10}}{3!} + \frac{x^{16}}{5!} - \dots$, $x^4 - \frac{x^{10}}{3!} + \frac{x^{16}}{5!} - \dots$ is the Taylor series of $f(x)$ at 0. This means that the coefficient of the x^n term is $\frac{f^{(n)}(0)}{n!}$ (because this is the formula we used to find the coefficients of the Taylor series).

So, $\frac{f'''(0)}{3!} = 0$ (because 0 is the coefficient of the x^3 term in $x^4 - \frac{x^{10}}{3!} + \frac{x^{16}}{5!} - \dots$) and $\frac{f^{(4)}(0)}{4!} = 1$ (because 1 is the coefficient of the x^4 term in $x^4 - \frac{x^{10}}{3!} + \frac{x^{16}}{5!} - \dots$). Solving, we get $\boxed{f'''(0) = 0}$ and $\boxed{f^{(4)}(0) = 4!}$.

A slower, but equally valid method: Another way you can tackle this problem is to start again with $f(x) = x^4 - \frac{x^{10}}{3!} + \frac{x^{16}}{5!} - \dots$ and just differentiate. We get:

$$\begin{aligned} f(x) &= x^4 - \frac{x^{10}}{3!} + \frac{x^{16}}{5!} - \dots \\ f'(x) &= 4x^3 - \frac{10x^9}{3!} + \frac{16x^{15}}{5!} - \dots \\ f''(x) &= 4 \cdot 3x^2 - \frac{10 \cdot 9x^8}{3!} + \frac{16 \cdot 15x^{14}}{5!} - \dots \\ f'''(x) &= 4 \cdot 3 \cdot 2x - \frac{10 \cdot 9 \cdot 8x^7}{3!} + \frac{16 \cdot 15 \cdot 14x^{13}}{5!} - \dots \\ f^{(4)}(x) &= 4 \cdot 3 \cdot 2 - \frac{10 \cdot 9 \cdot 8 \cdot 7x^6}{3!} + \frac{16 \cdot 15 \cdot 14 \cdot 13x^{12}}{5!} - \dots \end{aligned}$$

If we plug in $x = 0$ to the last two equations, we get $f'''(0) = 0$ and $f^{(4)}(0) = 4 \cdot 3 \cdot 2$.

Note: Either of these two methods should be a lot faster than just starting with $f(x) = x \sin(x^3)$ and differentiating that 4 times; this is one advantage of being able to write $f(x)$ as a power series. Of course, if you wanted to know $f^{(100)}(0)$, the first method is going to be a lot faster than the second.

3. (a) *Find a power series representation of $\arctan(5x)$ centered at 0.*

Solution. Let's start by finding a power series representation of $\arctan x$. Then we can replace x by $5x$ to get a power series representation of $\arctan(5x)$.

We know that the derivative of $\arctan x$ is $\frac{1}{1+x^2}$.

We start with:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \text{ valid when } |x| < 1.$$

Let's replace x by $-x^2$:

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-x^2)^n, \text{ valid when } |-x^2| < 1.$$

We'll simplify the right side a little:

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n}, \text{ valid when } |-x^2| < 1.$$

Note that $|-x^2| = |x^2| = |x|^2$, so saying $|-x^2| < 1$ is the same as saying $|x|^2 < 1$, or $|x| < 1$. That is, the radius of convergence of the power series on the right is 1.

Now integrate both sides:

$$\arctan x = C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}.$$

Integrating a power series doesn't change the radius of convergence, so the radius of convergence of this power series is still 1.

We need to solve for the constant of integration C ; we do this by plugging in $x = 0$ on both sides of the equation:

$$\arctan 0 = C + 0,$$

so $C = 0$. Therefore,

$$\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}.$$

Since the radius of convergence of this power series is 1, the power series converges when $|x| < 1$ and diverges when $|x| > 1$.

Finally, we replace x by $5x$:

$$\arctan 5x = \sum_{n=0}^{\infty} (-1)^n \frac{(5x)^{2n+1}}{2n+1}.$$

The power series converges when $|5x| < 1$ and diverges when $|5x| > 1$. It's nice to simplify this a little bit, so we end up with

$$\arctan 5x = \boxed{\sum_{n=0}^{\infty} (-1)^n \frac{5^{2n+1}}{2n+1} x^{2n+1}}.$$

(b) *What is the radius of convergence of the power series you found in part (a)?*

Solution. In part (a), we said that the power series converges when $|5x| < 1$ and diverges when $|5x| > 1$. In other words, the power series converges when $|x| < \frac{1}{5}$ and diverges when $|x| > \frac{1}{5}$, so the radius of convergence is $\boxed{\frac{1}{5}}$.

4. In each part, find a power series that has the given interval of convergence. (Hint: If you get stuck, try finding the interval of convergence of $\sum_{n=1}^{\infty} \frac{x^n}{n}$.)

(a) $(-6, 0)$.

Solution. We know the geometric series $\sum_{k=0}^{\infty} x^k$ converges when $|x| < 1$ and diverges when $|x| \geq 1$.

We're looking for something that converges when $|x+3| < 3$ and diverges when $|x+3| \geq 3$. Another way of saying this is that we want something that converges when $|\frac{x+3}{3}| < 1$ and diverges when

$|\frac{x+3}{3}| \geq 1$. The geometric series $\sum_{k=0}^{\infty} \left(\frac{x+3}{3}\right)^k$ works. (Of course, there are infinitely many other possible answers.)

(b) $(-1, 3)$.

Solution. Now we want something that converges when $|\frac{x+1}{2}| < 1$ and diverges when $|\frac{x+1}{2}| \geq 1$.

One possibility is the geometric series $\sum_{k=0}^{\infty} \left(\frac{x+1}{2}\right)^k$.

(c) *Challenge:* $[-1, 3)$.

Solution. This is more difficult because we can't use a geometric series. (The interval of convergence of a geometric series never includes its endpoints, but here we want to include the left endpoint.) Remember that the times we've had series where one endpoint is included but the other is not is when one endpoint gives us the alternating harmonic series (convergent) and the other gives us the harmonic series (divergent). So, let's try to make a power series where plugging in $x = -1$ gives us the alternating harmonic series and plugging in $x = 3$ gives us the harmonic series.

Using the hint, we'll look at $\sum_{n=1}^{\infty} \frac{x^n}{n}$. This has an interval of convergence of $[-1, 1)$, so it's almost what we want. Let's try to stretch and translate the function so that its interval of convergence will be $[-1, 3)$. First, we want to stretch it so that the radius of convergence is 2 instead of 1. To do this, we replace x by $\frac{x}{2}$: $\sum_{n=1}^{\infty} \frac{(x/2)^n}{n}$. Next, we want to shift the center to 1: to do this, we replace

x by $x - 1$: $\sum_{n=1}^{\infty} \frac{((x-1)/2)^n}{n}$, or $\sum_{n=1}^{\infty} \frac{(x-1)^n}{2^n \cdot n}$.