Differential Equations: An Introduction to Modeling

In #1 - #8, write a differential equation that reflects the situation. Include an initial condition if the information is given.

1. *The population of a certain country increases at a rate proportional to the population size. Let $P = P(t)$ be the population at time $t$.*

   **Solution.** The rate of change is $\frac{dP}{dt}$, and we also know that the rate of change is proportional to $P$, so it's $kP$ for some $k$. (We know $k$ must be positive because the population is increasing.) So, $\boxed{\frac{dP}{dt} = kP}$.

2. *A snowball melts at a rate proportional to its surface area. At time 0, the snowball has a radius of 10 cm. Let $r = r(t)$ be the radius of the snowball at time $t$.*

   **Solution.** The first sentence can be written as $\boxed{\frac{dr}{dt} = k(4\pi r^2)}$, where $k$ is a constant. (In this case, $k$ is going to be negative because the rate of change should be negative.) The second sentence can be written as $\boxed{r(0) = 10}$.

3. *A yellow rubber duck is dropped out of the window of an apartment building at a height of 80 feet. Let $s = s(t)$ be the height of the duck above the ground at time $t$. (Gravity is the acceleration $-32$ ft/s$^2$.)*

   **Solution.** The rubber duck accelerates due to gravity, so $\boxed{s''(t) = -32}$. We know that it starts at 80 feet above the ground, so $\boxed{s(0) = 80}$. We also know that the duck is not moving at the beginning, so its initial velocity is 0. That is, $\boxed{s'(0) = 0}$.

4. *Ferdinand is trying to fill a bucket from a faucet. Unfortunately, he doesn't realize that there is a small hole in the bottom of the bucket. Water flows in to the bucket from the faucet at a constant rate of .75 quarts per minute, and it flows out of the hole at a rate proportional to the amount of water $W(t)$ already in the bucket (due to the increased water pressure).*

   **Solution.** The rate of change of water in the bucket is $\frac{dW}{dt}$, which is equal to (rate of water coming in) minus (rate of water coming out). The rate of water coming in is 0.75. The rate of water going out is $kW(t)$ where $k$ is a positive constant. So, $\boxed{\frac{dW}{dt} = 0.75 - kW}$.

5. *A drug is being administered to a patient at a constant rate of $c$ mg/hr. The patient metabolizes and eliminates the drug at a rate proportional to the amount in his body. Let $M = M(t)$ be the amount (in mg) of medicine in the patient's body at time $t$, where $t$ is measured in hours.*

   **Solution.** The rate of change of medicine in the patient's body is equal to (rate in) minus (rate out). The rate in is $c$. The rate out is proportional to the amount in his body, so it's $kM$ for some positive constant $k$. Therefore, the appropriate model is $\boxed{\frac{dM}{dt} = c - kM}$.

6. *\$6000 is deposited in a bank account. The account has a nominal annual interest rate of 2%, compounded continuously. There are no deposits and no withdrawals. Let $M = M(t)$ be the amount of money in the account at time $t$, where $t$ is measured in years.*

**Solution.** If $t = 0$ is the time the money is deposited, then the first sentence is saying that $\boxed{M(0) = 6000}$. The second sentence is saying that $\boxed{\frac{dM}{dt} = .02M}$.

7. *\$6000 is deposited in a bank account. The account has a nominal annual interest rate of 2%, compounded continuously. Money is being withdrawn at a rate of \$500 per year.*[1] *Let $M = M(t)$ be the amount of money in the account at time $t$, where $t$ is measured in years.*

   **Solution.** The rate at which money is leaving the account is 500. Everything else is the same as in the previous problem, so we have $\boxed{\frac{dM}{dt} = .02M - 500}$ and $\boxed{M(0) = 6000}$.

8. *A rumor spreads at a rate proportional to product of the number of people who have heard it and the number who have not. In a town of $N$ people, suppose 1 person originates the rumor at time $t = 0$. Let $y = y(t)$ be the number of people who have heard the rumor at time $t$.*

   *What does this model imply about the number of people who eventually have heard the rumor?*

   **Solution.** The initial condition is $y(0) = 1$ since 1 person has heard the rumor at time 0. The rate of change is proportional to the product of the number of people who have heard it ($y$) and the number who have not ($N - y$), so $\boxed{\frac{dy}{dt} = ky(N - y)}$, where $k$ is a positive constant.

   As long as not everybody has heard the rumor, the spreading rate is positive. Therefore, it seems like eventually everybody will have heard the rumor.

The following problems are about solutions to differential equations.

9. *Which of the following is a solution to $\frac{dy}{dx} = y$?*

   *(a)* $y = \frac{x^2}{2} + C$.

   *(b)* $y = e^x + C$.

   *(c)* $y = Ce^x$.

   **Solution.** $\boxed{\text{(c)}}$. If $y = Ce^x$, then $\frac{dy}{dx} = Ce^x = y$.

   If $y = \frac{x^2}{2} + C$, then $\frac{dy}{dx} = x$, which is not equal to $y$.

   If $y = e^x + C$, then $\frac{dy}{dx} = e^x$, which is not equal to $y$ (unless $C = 0$).

10. *Give two solutions to $\frac{dy}{dx} = 5y$. What is the general solution?*

    **Solution.** A general solution is $\boxed{Ce^{5x}}$. Two specific solutions are $e^{5x}$ and $-e^{5x}$.

11. *Give two solutions to to $\frac{dy}{dx} = 5x$. What is the general solution?*

    **Solution.** We actually know how to solve this already: the equation says we are looking for a function of $x$ whose derivative is $5x$. That is, we want antiderivatives of $5x$, and we know that the general antiderivative is $\boxed{\frac{5}{2}x^2 + C}$. Two specific solutions are $\frac{5}{2}x^2$ and $\frac{5}{2}x^2 - 1$.

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[1] In reality, you cannot withdraw money continuously from the bank, but it's convenient to use a continuous model.