

## Separation of Variables / Mixing Problems

1. Find the general solution of the differential equation  $\frac{dM}{dt} = 2.4 - .2M$ . (Such a differential equation came up, for instance, when we modeled the amount of medicine in a patient's body.)

**Solution.** We can use separation of variables:  $\frac{1}{2.4-0.2M} dM = dt$ .<sup>1</sup> Simplifying,

$$-\frac{1}{0.2} \cdot \frac{1}{M-12} dM = dt.$$

Integrating both sides,

$$-\frac{1}{0.2} \ln |M-12| = t + C.$$

Multiplying both sides by  $-0.2$  gives

$$\ln |M-12| = -0.2t - 0.2C.$$

Since  $-0.2C$  is just an arbitrary constant, we can give it a new name; let's call it  $A$ . So,

$$\ln |M-12| = A - 0.02t.$$

Then,

$$M-12 = \pm e^A e^{-0.02t}.$$

Again,  $\pm e^A$  is just an arbitrary constant, so let's call it  $B$ . So,

$$M-12 = B e^{-0.02t},$$

and  $\boxed{M(t) = 12 + B e^{-0.02t}}$ .<sup>2</sup>

2. Last time, we solved the differential equation  $\frac{dy}{dt} = -\frac{t}{y}$  by drawing the slope field, guessing the solution, and checking it. Now, solve the differential equation using separation of variables.

**Solution.** We can rewrite  $\frac{dy}{dt} = -\frac{t}{y}$  as

$$y dy = -t dt.$$

Integrating both sides,

$$\frac{1}{2}y^2 = -\frac{1}{2}t^2 + C.$$

Multiplying both sides by 2,

$$y^2 = -t^2 + 2C.$$

Since  $2C$  is still just an arbitrary constant, we can give it a new name; let's call it  $A$ . So,

$$y^2 = A - t^2,$$

and  $\boxed{y = \pm \sqrt{A - t^2}}$ .

---

<sup>1</sup>Technically, we can only do this if  $2.4 - 0.2M \neq 0$ ; if  $2.4 - 0.2M = 0$ , which happens when  $M = 12$ , the original differential equation is just  $\frac{dM}{dt} = 0$ , so  $M(t) = 12$  is a solution.

<sup>2</sup>Since  $B = \pm e^A$ ,  $B$  should technically be non-zero. But we remarked earlier that  $M(t) = 12$  is a solution, so  $B = 0$  is also okay.

3. Solve the differential equation  $\frac{dy}{dt} = e^{-t-y}$ , and find the particular solution satisfying the initial condition  $y(0) = 1$ .

**Solution.** We can rewrite  $\frac{dy}{dt} = e^{-t-y}$  as

$$e^y dy = e^{-t} dt.$$

Integrating both sides,

$$e^y = -e^{-t} + C.$$

So,  $y(t) = \ln(C - e^{-t})$ . Plugging in the initial condition gives  $1 = \ln(C - 1)$ , so  $e = C - 1$ , and  $C = 1 + e$ . So, our answer is  $\boxed{y(t) = \ln(1 + e - e^{-t})}$ .

4. Solve the differential equation  $y' = 2y - 6$ .

**Solution.** First,  $y(t) = 3$  is a solution.

If  $y \neq 3$ , we can rewrite the differential equation as  $\frac{dy}{dt} = 2y - 6$ , or  $\frac{1}{2y-6} dy = dt$ . To integrate both sides, it's helpful to rewrite the left side as  $\frac{1}{2} \cdot \frac{1}{y-3}$ . So, we have

$$\begin{aligned} \int \frac{1}{2} \frac{1}{y-3} dy &= \int dt \\ \frac{1}{2} \ln|y-3| &= t + C \\ \ln|y-3| &= 2t + A \text{ where } A = 2C \\ |y-3| &= e^A e^{2t} \\ y-3 &= \pm e^A e^{2t} \\ y-3 &= B e^{2t} \text{ where } B = \pm e^A \\ y &= \boxed{3 + B e^{2t}} \end{aligned}$$

Here,  $B$  can be 0 (because we started out saying that  $y(t) = 3$  is a solution), or it can be  $\pm e^A$ , which accounts for any positive or negative constant. So,  $B$  can be any constant.

5. Which of the following differential equations are separable? (You need not solve.)

(a)  $\frac{dy}{dt} = t + y$ .

(b)  $\frac{dy}{dt} = \frac{y}{\sin t}$ .

(c)  $\frac{dy}{dt} = \frac{\sin t}{y} + t$ .

**Solution.** (a) is not separable.

(b) is separable, for we can rewrite it as  $\frac{1}{y} dy = \frac{1}{\sin t} dt$ .

(c) is not separable.

6. A 20-quart juice dispenser in a cafeteria is filled with a juice mixture that is 10% mango and 90% orange juice. A pineapple-mango blend (40% pineapple and 60% mango) is entering the dispenser at a rate of 4 quarts an hour and the well-stirred mixture leaves at a rate of 4 quarts an hour. Model the situation with a differential equation whose solution,  $M(t)$ , is the amount of mango juice in the container at time  $t$ . ( $t = 0$  is the time when the pineapple-mango blend starts to enter the dispenser.)

**Solution.** Since  $M(t)$  is the amount of mango juice in the container at time  $t$ ,  $\frac{dM}{dt}$  is the rate of change of the amount of mango juice in the container. We know that this is equal to (rate at which mango juice is entering the container) minus (rate at which mango juice is leaving the container).

Let's first focus on the stuff entering the container. This is a pineapple-mango blend, entering at a rate of 4 quarts per hour. Only 60% of this is mango juice though, so mango juice is entering the container at a rate of  $.6 \cdot 4 = 2.4$  quarts per hour.

Now, let's look at the stuff exiting the container. This is a blend of all of the juices, and it's leaving at a rate of 4 quarts per hour. What we need to know is what percent of this blend is mango juice. This is simple if we think about what our variables mean:  $M(t)$  is the amount of mango juice in the container at time  $t$ , while there is always 20 quarts of juice in all. So, the percentage of the 20 quarts which is mango juice is  $\frac{M(t)}{20}$ . Therefore, mango juice is leaving the container at a rate of  $\frac{M(t)}{20} \cdot 4 = \frac{M(t)}{5}$  quarts per hour.

So, our final differential equation is  $\frac{dM}{dt} = 2.4 - \frac{M}{5}$ . We also have an initial condition, since we know how much mango juice is in the dispenser at the beginning: 10% of the 20 quarts, or 2 quarts. So, our initial condition is  $M(0) = 2$ .

7. Suppose that, in the previous problem, the mixture was leaving at a rate of 5 quarts per hour rather than 4 quarts per hour. Model the new situation.

**Solution.** We still need to use  $\frac{dM}{dt} =$  (rate at which mango juice is entering the container) minus (rate at which mango juice is leaving the container), and the rate at which mango juice is entering the container is just like in the previous problem, 2.4 quarts per hour.

Let's look at the stuff exiting the container. Again, we need to use the formula (rate at which mango juice is leaving) = (concentration of mango juice in the mixture) times (rate at which mixture is leaving). The rate at which the mixture is leaving is 5 quarts per hour. The concentration of mango juice in the mixture is equal to (amount of mango juice in mixture) divided by (total amount of mixture). The amount of mango juice is exactly  $M(t)$ . The total amount of mixture is a little more complicated. At time  $t = 0$ , there is 20 quarts of juice in the container. However, because the juice is entering at a rate of 4 quarts per minute and leaving at 5 quarts per minute, there is a net loss of 1 quart per minute. Thus, after  $t$  minutes, the amount of juice in the container is  $20 - t$ . So, the concentration of mango juice in the mixture at time  $t$  is  $\frac{M(t)}{20-t}$ , and the rate at which mango juice is leaving the container is  $\frac{M(t)}{20-t} \cdot 5$ .

So, our final differential equation is  $\frac{dM}{dt} = 2.4 - \frac{5M}{20-t}$ . Again, we have the initial condition  $M(0) = 2$ .