Second-Order Homogeneous Differential Equations with Constant Coefficients

1. *Solve* $y'' - 6y' + 9y = 0$.

   Solution. The characteristic equation is $r^2 - 6r + 9 = 0$, or $(r-3)^2 = 0$. Since $r = 3$ is a repeated root of this equation, the general solution is $\boxed{C_1 e^{3t} + C_2 t e^{3t}}$.

2. *Solve* $y'' + y = 0$.

   Solution. The characteristic equation is $r^2 + 1 = 0$, and the roots are $r = \pm i$. Since $e^{it} = \cos t + i \sin t$ and $e^{-it} = \cos(-t) + i\sin(-t) = \cos t - i \sin t$, two different solutions are $\cos t$ and $\sin t$. Thus, the general solution is $\boxed{y(t) = C_1 \cos t + C_2 \sin t}$.

3. *Solve* $y'' - 2y' + 5y = 0$.

   Solution. The characteristic equation is $r^2 - 2r + 5 = 0$. The roots of this are

$$\begin{aligned}\frac{2 \pm \sqrt{2^2 - 4(1)(5)}}{2} &= \frac{2 \pm \sqrt{-16}}{2} \\ &= \frac{2 \pm 4i}{2} \\ &= 1 \pm 2i\end{aligned}$$

   So, we know two solutions are $e^{(1+2i)t}$ and $e^{(1-2i)t}$, and the general solution is $C_1 e^{(1+2i)t} + C_2 e^{(1-2i)t}$. We're only interested in the real solutions, so let's rewrite $e^{(1+2i)t}$ and $e^{(1-2i)t}$ to find the real solutions:

$$\begin{aligned}e^{(1+2i)t} &= e^{t+2it} \\ &= e^t \cdot e^{i(2t)} \\ &= e^t(\cos 2t + i \sin 2t) \\ &= e^t \cos 2t + i e^t \sin 2t\end{aligned}$$

   Similarly, $e^{(1-2i)t} = e^t \cos 2t - i e^t \sin 2t$.

   Our general solution was $C_1 e^{(1+2i)t} + C_2 e^{(1-2i)t}$, and we now know that we can rewrite this as $C_1 \left(e^t \cos 2t + i e^t \sin 2t\right) + C_2 \left(e^t \cos 2t - i e^t \sin 2t\right)$. Re-grouping the terms, we see that we can write this as $\boxed{A_1 e^t \cos 2t + A_2 e^t \sin 2t}$.

4. (a) *Solve* $y'' + 2y' + 4y = 0$ *with initial conditions* $y(0) = 1$ *and* $y'(0) = 0$.

   Solution. The characteristic equation $r^2 + 2r + 4 = 0$ has roots $r = \frac{-2 \pm \sqrt{4-16}}{2} = -1 \pm \sqrt{3}i$. Therefore, two different solutions are $e^{(-1+\sqrt{3}i)t}$ and $e^{(-1-\sqrt{3}i)t}$. We rewrite:

$$\begin{aligned}e^{(-1+\sqrt{3}i)t} &= e^{-t+\sqrt{3}ti} \\ &= e^{-t} \cdot e^{i(\sqrt{3}t)} \\ &= e^{-t}\left[\cos\left(\sqrt{3}t\right) + i \sin\left(\sqrt{3}t\right)\right]\end{aligned}$$

   Following the same reasoning as in the previous problem, we see that the general solution is $y(t) = C_1 e^{-t} \cos\left(\sqrt{3}t\right) + C_2 e^{-t} \sin\left(\sqrt{3}t\right)$.

   Let's use the initial conditions to solve for C_1 and C_2. The initial condition $y(0) = 1$ tells us $C_1 \cos(0) + C_2 \sin(0) = 1$. Since $\cos(0) = 1$ and $\sin(0) = 0$, we know that $C_1 = 1$.

To use the second initial condition, we first need to differentiate $y(t)$:

$$y'(t) = C_1\left[-e^{-t}\cos\left(\sqrt{3}t\right) - \sqrt{3}e^{-t}\sin\left(\sqrt{3}t\right)\right] + C_2\left[-e^{-t}\sin\left(\sqrt{3}t\right) + \sqrt{3}e^{-t}\cos\left(\sqrt{3}t\right)\right].$$

Therefore, $y'(0) = -C_1 + \sqrt{3}C_2$. So, $0 = -1 + \sqrt{3}C_2$, and $C_2 = \frac{1}{\sqrt{3}}$.

Thus, our final solution is $\boxed{y(t) = e^{-t}\cos\left(\sqrt{3}t\right) + \frac{1}{\sqrt{3}}e^{-t}\sin\left(\sqrt{3}t\right)}$.

(b) *Interpret part (a) in terms of a vibrating spring. What is happening to the spring as time goes on?*

Solution. The differential equation $y'' + 2y' + 4y = 0$ describes a vibrating spring with friction. The initial condition $y(0) = 1$ says that the spring is initially stretched 1 unit beyond its equilibrium position, and the initial condition $y'(0) = 0$ says that its initial velocity is 0.

Our solution was $y(t) = e^{-t}\cos\left(\sqrt{3}t\right) + \frac{1}{\sqrt{3}}e^{-t}\sin\left(\sqrt{3}t\right)$, or $y(t) = e^{-t}\left[\cos\left(\sqrt{3}t\right) + \frac{1}{\sqrt{3}}\sin\left(\sqrt{3}t\right)\right]$. This function oscillates while decreasing in magnitude, and $\lim_{t\to\infty} y(t) = 0$.

5. *Which of the following differential equations has periodic solutions? What is the period?*

(a) $y'' + 2y' - 3y = 0$.

(b) $y'' + 2y + 3y = 0$.

(c) $y'' + 4y' = 0$.

(d) $y'' + 4y = 0$.

(e) $y'' - 4y = 0$.

Does this agree with your interpretation of the differential equations in terms of vibrating springs?

Solution. Let's look at each one.

(a) The characteristic equation is $r^2 + 2r - 3 = 0$, or $(r+3)(r-1) = 0$. So, the general solution is $C_1e^{-3t} + C_2e^t$, which is not periodic. This differential equation can't be interpreted in terms of vibrating springs ($y'' + by' + cy = 0$ is only the differential equation for a vibrating spring if $b \geq 0$ and $c > 0$).

(b) The characteristic equation is $r^2 + 2r + 3 = 0$, which has roots $r = \frac{-2\pm\sqrt{-8}}{2} = -1 \pm \sqrt{2}i$. Since $e^{(-1+\sqrt{2}i)t} = e^{-t} \cdot e^{i(\sqrt{2}t)} = e^{-t}(\cos\sqrt{2}t + i\sin\sqrt{2}t)$, if we follow the reasoning we used in #3, we find that the general solution is $C_1e^{-t}\cos(\sqrt{2}t) + C_2e^{-t}\sin(\sqrt{2}t)$. This is not periodic. This differential equation describes a vibrating spring subject to friction (the fact that $b > 0$ means there is friction), so it makes sense that the solution is not periodic. (Such a spring should vibrate less and less over time rather than vibrating the same amount forever.)

(c) The characteristic equation is $r^2 + 4r = 0$, or $r(r+4) = 0$. This has roots $r = 0$ and $r = -4$, so the general solution is $C_1 + C_2e^{-4t}$, which is not periodic. This differential equation can't be interpreted in terms of vibrating springs.

(d) The characteristic equation is $r^2 + 4 = 0$, or $r = \pm 2i$. Since $e^{2it} = \cos 2t + i\sin 2t$, if we follow the reasoning we used in #3, we find that the general solution is $C_1\cos(2t) + C_2\sin(2t)$. This is periodic with period π. This differential equation describes a vibrating spring without friction (since $b = 0$), and it makes sense that such a spring should oscillate back and forth periodically.

(e) The characteristic equation is $r^2 - 4 = 0$, or $(r-2)(r+2) = 0$. This has roots $r = 2$ and $r = -2$, so the general solution is $C_1 e^{-2t} + C_2 e^{-2t}$, which is not periodic. This differential equation can't be interpreted in terms of vibrating springs.