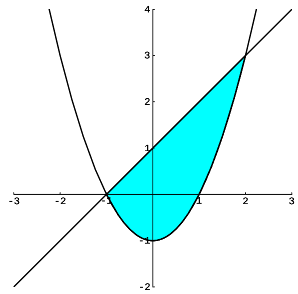
The Definite Integral

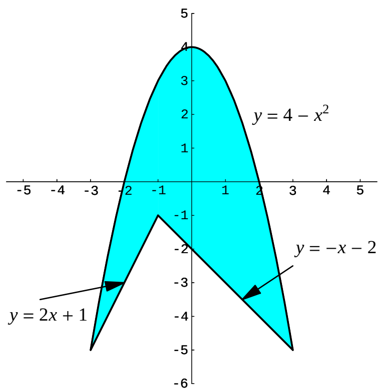
1. The definite integral is *defined* to be

   (a) a limit of Riemann sums.
   (b) the difference in the evaluation of an antiderivative at the endpoints of the interval.
   (c) a signed area.
   (d) all of the above.

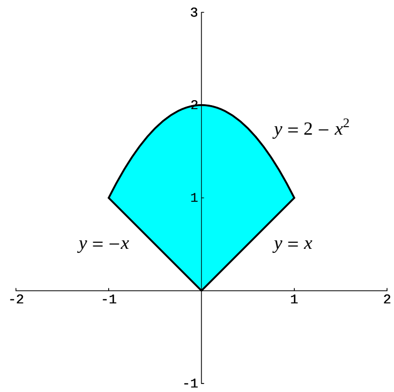
2. Write an integral or a sum and/or difference of integrals that gives the area enclosed by the graphs of $y = x^2 - 1$ and $y = x + 1$.



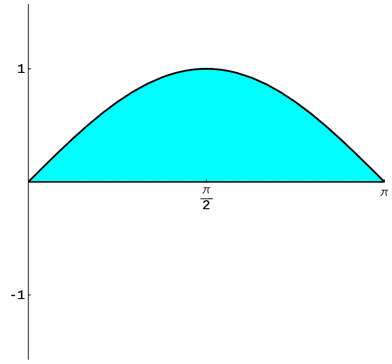
3. Find the area of the following region.



1. Find the area of the region shown. (You may leave your answer as an integral or sum/difference of integrals.)



2. Here is one loop of the sine curve.

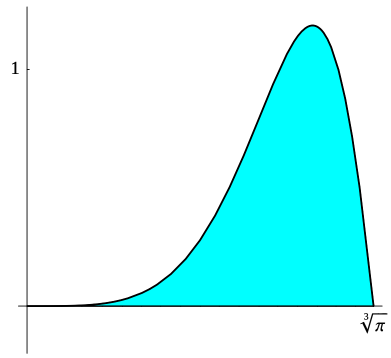


(a) If you rotate this region about the x-axis, what shape do you get? What is its volume? (You do not need to evaluate your integral.)

(b) If you rotate the region about the line $y = -1$, what shape do you get? What is its volume? (You do not need to evaluate your integral.)

More on Volumes

1. This is the curve $y = x \sin x^3$. If we rotate this region about the y-axis, what is the volume of the resulting solid? (Once you get an integral, try to evaluate it.)



2. How can you describe a bagel as a solid of revolution? (That is, what sort of region would you rotate, and what line would you rotate it about?)

3. The disk of radius 3 centered at the origin is rotated about the line $x = 4$. Find the volume using vertical slices. (It is also possible to do it using horizontal slices, and you might want to try that for extra practice.)

4. Let $\mathcal{R}$ be the region enclosed by the x-axis, the y-axis, $y = 1$, and $y = \frac{1}{x} - 1$.

   (a) Find the volume generated when $\mathcal{R}$ is rotated about the line $x = -2$.

   (b) Find the volume generated when $\mathcal{R}$ is rotated about the line $y = 2$.

Integration by Parts

Evaluate the following integrals.

1. $\int xe^x \; dx.$

2. $\int x \ln x \; dx.$

3. $\int_1^e \ln x \; dx.$

4. $\int_0^1 \arctan x \; dx.$

5. $\int x^2 \cos 2x \; dx.$

6. $\int e^x \cos x \; dx.$

7. $\int \cos \sqrt{x} \; dx.$

8. You are given the following information about an unknown function $g(x)$:

$$\int_1^2 \frac{g(u)}{u} \; du = 3, \int_1^2 g(u) \; du = 4, \int_1^4 g(u) \; du = 5, g(1) = 2, g(2) = -2.$$

(a) Evaluate $\int_1^2 (\ln x) g'(x) \; dx.$

(b) Evaluate $\int_1^2 x g(x^2) \; dx.$

9. $\int \sin 5x \sin 3x \; dx.$

Partial Fractions

1. Which of the following is easiest to integrate?

   (a) $\displaystyle\int \frac{5x-4}{x^2-x-2}\,dx.$

   (b) $\displaystyle\int \frac{5x-4}{(x-2)(x+1)}\,dx.$

   (c) $\displaystyle\int \frac{5x}{x^2-x-2}\,dx - \int \frac{4}{x^2-x-2}\,dx.$

   (d) $\displaystyle\int \frac{3}{x+1}\,dx + \int \frac{2}{x-2}\,dx$

   How do the four choices relate to each other?

2. Evaluate the following integrals.

   (a) $\displaystyle\int \frac{1}{y^2-4}\,dy.$

   (b) $\displaystyle\int \frac{5x-7}{x^2-3x+2}\,dx.$

3. Write down the form of the partial fraction expansion for the following integrals. (You don't need to actually solve for the coefficients.)

(a) $\displaystyle\int \frac{3x^2 + x + 5}{(x+1)(x+3)(x-5)}.$

(b) $\displaystyle\int \frac{x+1}{(x^2+4)(x^2+9)}.$

(c) $\displaystyle\int \frac{x^3 + 2x}{(x+4)(x+3)(x+2)^2}.$

(d) $\displaystyle\int \frac{x^2 + 1}{(x+1)^2(x^2+5)}.$

(e) $\displaystyle\int \frac{3x^2}{x^2 + 2x + 1}\, dx.$

(f) $\displaystyle\int \frac{x^3 + 4x^2 + 7x}{x^2 + 4x + 3}\, dx.$

4. Evaluate the following integrals.

(a) $\displaystyle\int \frac{x^2 - x + 4}{x^3 + 4x}.$

(b) $\displaystyle\int \frac{1}{x^3 + x}\, dx.$

(c) $\displaystyle\int \frac{\sin\theta \; d\theta}{\cos^2\theta + \cos\theta - 2}.$

Integration Techniques

In each problem, decide which method of integration you would use. If you would use substitution, what would u be? If you would use integration by parts, what would u and dv be? If you would use partial fractions, what would the partial fraction expansion look like? (Don't solve for the coefficients.)

1. $\displaystyle\int \frac{\cos x \; dx}{\sqrt{1+\sin x}}$.

2. $\displaystyle\int (\ln x)^2 \; dx$.

3. $\displaystyle\int e^x \sin x \; dx$.

4. $\displaystyle\int \frac{x}{x^2-1} \; dx$.

5. $\int xe^{x^2}\ dx.$

6. $\int \frac{x^2}{x^2+4x+3}\ dx.$

7. $\int \frac{e^t}{1+e^t}\ dt.$

8. $\int \arcsin x\ dx.$

Density and Slicing

1. A seaside village, Playa del Carmen, is in the shape of a rectangle 4 miles wide and 6 miles long. The sea lies along a 6-mile long side. People prefer to live near the water, so the density of people is given by $\rho(x) = 10000 - 800x$ people per square mile, where x is the distance from the seaside. We would like to find the population of the village.

   (a) Show in a sketch how to slice up the region.

   (b) What is the area of the k-th slice?

   (c) What is the approximate population in the k-th slice?

   (d) Write a general Riemann sum to estimate the total population of the city.

   (e) Find a definite integral expressing the population of the village.

2. A rectangular plot of farm land is 300 meters by 200 meters. A straight irrigation pipe 300 meters long runs down the center of the plot, dividing it in half lengthwise. The farmer's yield decreases as the distance from the irrigation pipe increases. Suppose that the yield is given by $\rho(x)$ grams per square meter, where x is the distance in meters from the irrigation pipe. Write an integral giving the total yield from the plot.

3. People in the Boston area like to live near the city center, so the population density around Boston is $\rho(r) = \frac{36{,}000}{r^2+2r+1}$ people per square mile, where r is the distance in miles to the center of Boston. Find the number of people who live within 5 miles of the center of Boston.

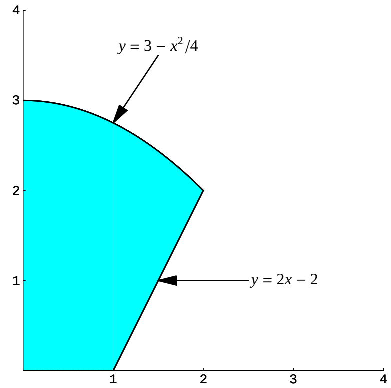
More Applications of Integration

1. A cone with height 8 inches and radius 6 inches is filled with flavored slush. When the cup is held upright with the pointed end resting on a table, the density of flavoring syrup in the cup varies with height above the table. Suppose $\rho(x)$ gives the number of ounces of syrup per cubic inch, where x is the distance from the table top. Write an integral giving the total amount of syrup in the cup.

2. Suppose the density of a planet is given by the function $\rho(r) = \dfrac{40000}{1 + 0.0001r^3}$ kilograms per cubic kilometer, where r is the distance in kilometers from the center of the planet. Find the total mass of the planet if its radius is 8000 km. (You do not need to evaluate your integral.)

3. A cylindrical candle of height 50 mm and radius 12 mm is formed by repeatedly dipping a wick of radius 1 mm into hot wax and then allowing the new layer of wax to dry. The density of each new layer is slightly different, so the density of the candle varies with the distance to the wick. If $\rho(x)$ gives the density in grams per cubic mm of the wax, where x measures the distance to the wick, write an integral giving the mass of the candle.

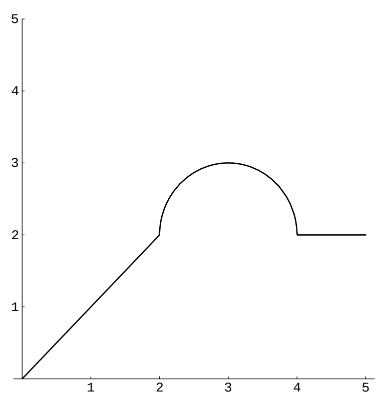
4. We can model a muffin as a solid of revolution, obtained by rotating the following region about the y-axis. Due to a poor recipe, the chocolate chips in our muffin tend to sink to the bottom. The amount of chocolate in the muffin is given by $\rho(y) = 5 - y$ grams per cubic inch, where y represents the distance to the bottom of the muffin. Find the total amount of chocolate in the muffin.



5. You ride your bike with velocity $v(t) = 3t^2 + 4t - 5$ in the time interval $[0, 3]$. What is your average velocity?

6. The temperature outside is given by the function $f(t)$, where t represents the time since 10:00 am. How would you find the average temperature between noon and 5:00 pm?

7. The graph of a function f is shown. The graph is made up of lines and semicircles. Find the average value of f on the interval $[1, 5]$.



Arc Length and Improper Integrals

1. Write an integral that gives the length of one arch of the sine curve (so from $x = 0$ to $x = \pi$).

2. (a) Does $\int_1^\infty \frac{1}{x^2}\,dx$ converge or diverge? If it converges, evaluate it.

   (b) Does $\int_1^\infty \frac{1}{x}\,dx$ converge or diverge? If it converges, evaluate it.

3. Using #2, can you conclude anything about whether the following integrals converge or diverge? (Try to figure this out without evaluating the integrals!)

   (a) $\displaystyle\int_1^\infty \frac{1}{x^3}\,dx$?

   (b) $\displaystyle\int_1^\infty \frac{1}{x^{1/2}}\,dx$?

   (c) $\displaystyle\int_1^\infty \frac{1}{x^{3/2}}\,dx$?

Improper Integrals

An integral is improper if

1. the interval of integration is unbounded and/or
2. the integrand is unbounded somewhere on the interval of integration

The basic strategy for dealing with an improper integral is:

1. Identify all improprieties.
2. If necessary, split up the integral into a sum so that every impropriety is an endpoint of integration and each piece of the sum has at most one impropriety.
3. Compute each of these improper integrals as a limit of definite integrals.
4. If any one summand (piece) diverges, the whole thing diverges. If every piece converges, sum to find out what the original converges to.

   ↯ **Caution**: This means that divergent summands can never "cancel" one another!

Determine whether the following integrals converge or diverge. Explain your reasoning.

1. $\displaystyle\int_{-2}^{2} \frac{x}{x^2-1}\,dx.$

2. $\displaystyle\int_{1}^{\infty} \frac{1}{x^4+2}\,dx.$

3. $\displaystyle\int_0^\infty \frac{1}{e^x + x}\ dx.$

4. $\displaystyle\int_{-\infty}^\infty \sin x\ dx.$

5. $\displaystyle\int_1^\infty \frac{1 + e^{-x}}{x}\ dx.$

6. $\displaystyle\int_1^\infty \frac{\cos^2 x}{x^2}\ dx.$

Probability

Waiting times, shelf-lifes, and equipment failure times are often modeled by exponentially decreasing probability density functions.

1. Suppose $f(t) = 0$ for $t < 0$ and $f(t) = 0.5e^{-ct}$ for $t \geq 0$ is the probability density function for the lifetime of a particular toy (t in years).

   (a) For what value of c is this a probability density function?

   (b) What is the probability that the toy lasts over one year? (Is there any way to compute this without computing an improper integral?)

   (c) What is the median life of this type of toy?

2. A large number of students take an exam. 30% of the students receive a score of 70, 50% receive a score of 80, and 20% receive a score of 90. What is the average score on the exam?

3. The density function for the duration of telephone calls within a certain city is $p(x) = 0.4e^{-0.4x}$ where x denotes the duration in minutes of a randomly selected call.

   (a) What percentage of calls last one minute or less?

   (b) What percentage of calls last between one and two minutes?

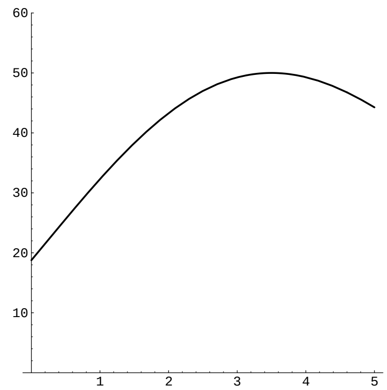
   (c) What percentage of calls last 3 minutes or more?

   (d) What is the average length of a call?

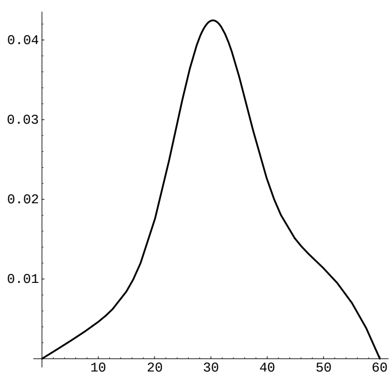
4. The lifetime, in hundreds of hours, of a certain type of light bulb has been found empirically to have a probability density function approximated by $f(x) = \frac{\sqrt{65}}{8(1+x^2)^{3/2}}$ for $0 < x < 8$. Find the mean lifetime of such a bulb.

More Probability

1. (a) The following function represents the temperature outside as a function of time. Estimate the average temperature between time 0 and time 5.



(b) A meteorologist takes several temperature readings which are described by the following probability density function. Estimate the average temperature.



2. Let $p(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$, a probability density function.

   (a) Sketch the graph of this probability density function. What do you think its mean is?

   (b) Verify your guess mathematically.

3. The bell curve with mean 0 and standard deviation s is given by the probability density function $p(x) = \frac{1}{s\sqrt{2\pi}} e^{-x^2/(2s^2)}$. What fraction of the population is within one standard deviation s of the mean 0?

Taylor Series

Last time, we learned how to approximate functions like $\cos x$ by polynomials. We called these polynomials Taylor polynomial approximations.

1. Find the degree 6 Taylor polynomial approximation for $f(x) = \sin x$ centered at 0.

2. (a) If you want to find a Taylor polynomial approximation $a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$ (centered at 0) to $f(x)$, write a formula for the coefficient a_k.

   (b) If you want to find a Taylor polynomial approximation $a_0 + a_1(x-3) + a_2(x-3)^2 + \cdots + a_n(x-3)^n$ (centered at 3) to $f(x)$, write a formula for the coefficient a_k.

We saw last time that we could make our approximations better by using higher and higher degree polynomials. As you guessed, our goal is to take the limit as the degree goes to infinity and write down "polynomials of infinite degree." Such things are called power series. They can be written as $a_0 + a_1(x-b) + a_2(x-b)^2 + a_3(x-b)^3 + \cdots$, or more compactly in summation notation as $\sum_{k=0}^{\infty} a_k(x-b)^k$. ($a_k$ and b are constants: a_k are the coefficients, and b is where you center.)

3. How do you think you would represent $\sin x$ as an infinite polynomial centered at 0? This is called the Taylor series (rather than Taylor polynomial) generated by $\sin x$ about 0.

4. What is the Taylor series generated by $\cos x$ about 0?

5. What is the Taylor series generated by e^x about 0?

6. We hope that, by using "polynomials of infinite degree," we end up with something that is not just an approximation for our function but is actually equal to the function. We don't really know if this is true yet. Taking on faith that e^x is actually equal to its Taylor expansion about 0, can you write a power series expansion (or "infinite polynomial representation") of:

   (a) e^{-x^2}?

   (b) $\int e^{-x^2}\, dx$?

7. (a) Write a general formula for the Taylor series of $f(x)$ centered at 0.

   (b) What if you wanted to center at 5?

Geometric Sums and Geometric Series

1. In your quest to become a millionaire by age 50, you start an aggressive savings plan. You open a new investment account on January 1, 2008 and deposit \$9000 into it every year on January 1. Each year, you earn 7% interest on December 31.

   (a) How much money will you have in your account on January 2, 2009? 2010? 2014? (Don't try to add or multiply things out; just write an arithmetic expression.)

   (b) Will you be a millionaire by age 50?

2. If you suffer from allergies, your doctor may suggest that you take Claritin once a day. Each Claritin tablet contains 10 mg of loratadine (the active ingredient). Every 24 hours, about 7/8 of the loratadine in the body is eliminated (so 1/8 remains). [1]

   (a) If you take one Claritin tablet every morning for a week, how much loratadine is in your body right after you take the 3rd tablet? 7th tablet? (Don't try to simplify your computations; just write out an arithmetic expression.)

[1] This estimate comes from the fact that the average half-life of loratadine is known to be 8 hours.

(b) If you take Claritin for years and years, will the amount of loratadine in your body level off? Or will your bloodstream be pure loratadine?

3. For what values of x does the geometric series $1 + x + x^2 + \cdots$ converge? [2] If it converges, what does it converge to?

4. Which of the following series are geometric?

(a) $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k 2^{k+1}}{3^k}$.

(b) $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^3}$.

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{2}{3^{n/2}}$.

[2] We could also write this series in summation notation as $\displaystyle\sum_{k=0}^{\infty} x^k$.

Let $a_1 + a_2 + a_3 + \cdots = \sum_{k=1}^{\infty} a_k$ be an infinite series, and let s_n denote its n-th partial sum: $s_n = a_1 + a_2 + a_3 + \cdots + a_n$.

- If $\lim_{n\to\infty} s_n = s$ for a finite number s, we say the series $a_1 + a_2 + a_3 + \cdots$ converges (to s) and write $a_1 + a_2 + a_3 + \cdots = s$.
- If $\lim_{n\to\infty} s_n$ does not exist (or is not finite), we say the series $a_1 + a_2 + a_3 + \cdots$ diverges.

1. Suppose you know that the infinite series $a_1 + a_2 + a_3 + \cdots + a_n + \cdots$ converges to s and that $a_k > 0$ for k any positive integer. Let $s_n = a_1 + a_2 + a_3 + \cdots + a_n$. For each of the following statements, determine whether the statement must be true, could possibly be true, or must be false.

   (a) $\lim_{n\to\infty} a_n = 0$.

   (b) $\lim_{n\to\infty} s_n = 0$.

   (c) There exists a number M such that $s_n < M$ for all n. (This is equivalent to saying that the partial sums are bounded. Why?)

   (d) $\sum_{k=5}^{\infty} a_k$ converges.

2. Suppose you know that $\lim_{n\to\infty} b_n = 0$. Can you be sure that the infinite series $b_1+b_2+b_3+\cdots$ converges?

A sequence is an ordered list of numbers. Some examples are:

- $-2, -4, -6, -8, \ldots$
- $-\frac{1}{2}, \frac{1}{4}, -\frac{1}{8}, \frac{1}{16}, \ldots$

Sometimes sequences are also written in the form $\{s_n\}$ where s_n is the n-th term of the sequence. For instance, the two examples above are $\{-2n\}$ and $\left\{\left(-\frac{1}{2}\right)^n\right\}$.

3. (a) Give an example of a sequence (ordered list) of numbers such that the numbers are increasing but are bounded.

   (b) Give an example of a sequence (ordered list) of numbers such that the numbers are increasing and are not bounded.

   (c) Give an example of a sequence (ordered list) of numbers such that the numbers are bounded but have no limit as $n \to \infty$.

A sequence is monotonic if it is either increasing (each term is greater than or equal to the preceding term) or decreasing (each term is less than or equal to the preceding term).

4. (a) A sequence which is both monotonic and bounded

   must converge   could either converge or diverge   must diverge

   (b) A sequence which is monotonic but not bounded

   must converge   could either converge or diverge   must diverge

5. Consider the series $\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$ (called the harmonic series).

   (a) Does the sequence of terms converge? If so, to what does it converge?

   (b) Does the sequence of partial sums converge? If so, to what does it converge?

   (c) Does the series $\sum_{k=1}^{\infty} \frac{1}{k}$ converge?

   (d) Does the series $\sum_{k=10^{10}}^{\infty} \frac{1}{100000000k}$ converge?

6. See if you can determine whether each of the following series converges or diverges by using the Nth Term Test for Divergence, results about geometric series, or some sort of comparison to series you know about.

   (a) $\sum_{k=100}^{\infty} \frac{1}{3k}$.

   (b) $\sum_{k=3}^{\infty} \frac{(-1)^k 2^k}{3^k}$.

   (c) $\sum_{k=4}^{\infty} \frac{(-1)^k 3^k}{2^k}$.

   (d) $\sum_{n=1}^{\infty} \frac{\ln n}{n}$.

   (e) $\sum_{n=2}^{\infty} \frac{n}{\ln n}$.

   (f) $\sum_{n=0}^{\infty} \sin n$.

Comparison

1. Use the Comparison Test (also known as "direct comparison") to decide whether the following series converge or diverge.

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}3^n}$.

   (b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n!}$.

   (c) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{n^3 - n + 1000}$.

   (d) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\ln(1+n)}$.

2. True or false: If $\{a_n\}$ is a sequence with positive terms and $\lim_{n\to\infty} a_n = 0$, then there is a number k such that $a_n < 1$ whenever $n \geq k$.

3. Decide whether the following series converge or diverge using any method you like.

(a) $\displaystyle\sum_{n=100}^{\infty} \cos n.$

(b) $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k 2^{k+1}}{3^k}.$

(c) $1 + 0 + (-1) + 1 + 0 + (-1) + 1 + 0 + (-1) + \cdots.$

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n}.$

More Comparison

1. For what values of p is the <u>p-series</u> $\sum_{n=1}^{\infty} \frac{1}{n^p}$ convergent?

2. Does the series $\sum_{n=2}^{\infty} \frac{1}{n^2 - 1}$ converge or diverge?

3. Does the series $\sum_{k=1}^{\infty} \frac{3}{8^k - 2}$ converge or diverge?

4. Otto is given the following problem for homework.

   Decide whether the series $\sum_{n=1}^{\infty} \sin^2(\pi n)$ converges or diverges. Explain your reasoning.

   Otto writes

   The improper integral $\int_1^{\infty} \sin^2(\pi x)\, dx$ diverges, so $\sum_{n=1}^{\infty} \sin^2(\pi n)$ also diverges by the Integral Test.

   Otto is correct that the improper integral diverges (although he should have shown more work!). But the rest of his reasoning is incorrect — why? And what is the correct answer to the problem?

5. Let $\sum a_n$ and $\sum b_n$ be series with positive terms. The Limit Comparison Test only applies when $\lim_{n\to\infty} \frac{a_n}{b_n}$ is a positive real number.

   (a) If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$, can you draw any conclusions?

   (b) If $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$, can you draw any conclusions?

Absolute and Conditional Convergence

1. Does the series $\sum_{k=1}^{\infty}(-1)^{k+1}\frac{1}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots$ converge or diverge? (This series is often called the alternating harmonic series.)

2. In fact, $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots = \ln 2$. Write a finite sum which estimates $\ln 2$ with error of less than 0.001. Is your approximation too big or too small?

Definition. *An <u>alternating series</u> is a series whose terms are alternately positive and negative. An alternating series can be written as either*

$$\begin{aligned} a_1 - a_2 + a_3 - \cdots &= \sum_{k=1}^{\infty}(-1)^{k+1}a_k \\ \text{or} \quad -a_1 + a_2 - a_3 + \cdots &= \sum_{k=1}^{\infty}(-1)^{k}a_k \end{aligned}$$

with all $a_k > 0$. (Which one depends on whether the first term is positive or negative.)

Definition. *A series $\sum a_n$ is called <u>absolutely convergent</u> if the series of absolute values $\sum |a_n|$ is convergent.*

3. Is the alternating harmonic series $\sum_{k=1}^{\infty}(-1)^{k+1}\frac{1}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots$ absolutely convergent?

Theorem. *If the series $\sum a_n$ is absolutely convergent, then it is convergent.*

The opposite is not true: there are convergent series which are not absolutely convergent. These series are called conditionally convergent.

4. Determine whether each series converges or diverges. If it converges, does it converge absolutely or conditionally?

   (a) $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k}{\sqrt{k}}$.

   (b) $\displaystyle\sum_{k=1}^{\infty} \frac{\sin k}{k!}$.

   (c) $\displaystyle\sum_{k=1}^{\infty} (-2)^k$.

5. (a) The Taylor series for $\cos x$ about 0 is $1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \displaystyle\sum_{k=0}^{\infty} (-1)^{k+1} \frac{x^{2k}}{(2k)!}$. Show that, if you plug in any value of x with $-0.5 \leq x \leq 0.5$, the series converges.

   (b) In fact, the series converges for all x, and $\cos x$ is actually equal to the series; that is,

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots.$$

   Suppose you use the approximation $\cos x \approx 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$ to approximate $\cos x$ when $-0.5 \leq x \leq 0.5$. Find an upper bound for the error. (This means: find a number U that you can show is bigger than the error.)

Ratio Test

1. What does the Ratio Test tell you about the following series?

   (a) $\displaystyle\sum_{k=0}^{\infty}(-1)^{k+1}\frac{1000^k}{k!}$.

   (b) $\displaystyle\sum_{k=1}^{\infty}\frac{1}{k}$.

   (c) $\displaystyle\sum_{k=1}^{\infty}\frac{1}{k^2}$.

2. When we studied Taylor series, we found that the Taylor series for $\sin x$ about 0 was $x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots$, which can be written in summation notation as $\displaystyle\sum_{k=0}^{\infty}(-1)^k\frac{x^{2k+1}}{(2k+1)!}$. For what values of x does this series converge?

3. When we studied Taylor series, we found that the Taylor series for $\ln(1+x)$ about 0 was $x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots$, which can be written in summation notation as $\displaystyle\sum_{k=1}^{\infty}(-1)^{k+1}\frac{x^k}{k}$. For what values of x does this series converge?

4. Decide whether the following series converge absolutely, converge conditionally, or diverge. You may use any method you like, but explain your reasoning. There is one that you will not be able to do (this is not due to a personal failing; it's just that all of the tests that we know are inconclusive).

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{\cos n}{n^2}.$

(b) $\displaystyle\sum_{n=100}^{\infty} \frac{n!n!}{(2n)!}.$

(c) $\displaystyle\sum_{n=0}^{\infty} \frac{\sin n}{n}.$

(d) $\displaystyle\sum_{n=2}^{\infty} \frac{\ln n}{n}.$

(e) $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{n^3+1}.$

(f) $\displaystyle\sum_{n=5000}^{\infty} (-1)^{n} \frac{n}{n+1}.$

Power Series

A power series centered at the number a is a series of the form $\sum_{n=0}^{\infty} c_n(x-a)^n$ where x is a variable and the c_n are constants.

1. For what values of x does the power series $\sum_{n=1}^{\infty} n!x^n$ converge? (This series is centered at 0.)

Theorem. For a given power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ centered at a, there are 3 possibilities:

1. The series converges only when $x = a$.
2. The series converges for all x.
3. There is a positive number R such that the series converges when $|x-a| < R$ and diverges when $|x-a| > R$. R is called the radius of convergence. (Note that this doesn't say anything about what happens when $|x-a| = R$.)

The interval of convergence of a power series is the set of x for which the power series converges.

2. Find the radius of convergence and interval of convergence of the power series $\sum_{n=1}^{\infty} \frac{2^n}{n}(x-3)^n$.

3. We know that the power series $\sum_{n=0}^{\infty} x^n$ converges to $\frac{1}{1-x}$ when $|x| < 1$. Find a power series representation of the function $\frac{x}{1+4x^2}$. What is the radius of convergence of this power series?

Theorem. If the power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ has radius of convergence R where $R > 0$ or $R = \infty$, then the function $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$ is differentiable on the interval $(a-R, a+R)$ and

1. $f'(x) = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1}$.

2. $\int f(x)\, dx = C + \sum_{n=0}^{\infty} \frac{c_n}{n+1}(x-a)^{n+1}$.

The power series in (1) and (2) both have radius of convergence R. (Note: Although the radius of convergence remains unchanged, the interval of convergence may change.)

4. (a) Find a power series representation for $\ln(1+x)$ centered at 0. What is the radius of convergence for the power series you have found? (Hint: What is the derivative of $\ln(1+x)$?)

   (b) Find the degree 5 Taylor polynomial approximation of $\ln(1+x)$.

5. Find a power series representation of $\arctan(5x)$ centered at 0. What is the radius of convergence of the power series you have found?

More on Power Series

1. Suppose we have a power series $\sum_{n=1}^{\infty} c_n(x+7)^n$.

   (a) If you know that the power series converges when $x = 0$, what conclusions can you draw?

   (b) Suppose you also know that the power series diverges when $x = 1$. Now what conclusions can you draw?

   (c) Does $\sum_{n=1}^{\infty} c_n$ converge (assuming that the power series converges when $x = 0$ and diverges when $x = 1$)?

   (d) Does $\sum_{n=1}^{\infty} c_n(-8.1)^n$ converge?

   (e) Does $\sum_{n=1}^{\infty} c_n(-8)^n$ converge?

Theorem. If $f(x)$ has a power series representation centered at a (that is, $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$ for $|x-a| < R$), then that power series must be the Taylor series of f at a.

2. (a) Taking for granted that $\sin x = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!}$ for all x, find the Taylor series of $x\sin(x^3)$ at 0.

   (b) What is the radius of convergence of the power series you found in part (a)?

   (c) Let $f(x) = x\sin(x^3)$. What is $f'''(0)$? $f^{(4)}(0)$?

Theorem. If the power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ has radius of convergence R where $R > 0$ or $R = \infty$, then the function $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$ is differentiable on the interval $(a - R, a + R)$ and

(a) $f'(x) = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1}$.

(b) $\int f(x)\, dx = C + \sum_{n=0}^{\infty} \frac{c_n}{n+1}(x-a)^{n+1}$.

The power series in (1) and (2) both have radius of convergence R. (Note: Although the radius of convergence remains unchanged, the interval of convergence may change.)

3. (a) Find a power series representation of $\arctan(5x)$ centered at 0.

   (b) What is the radius of convergence of the power series you found in part (a)?

4. In each part, find a power series that has the given interval of convergence. (Hint: If you get stuck, try finding the interval of convergence of $\sum_{n=1}^{\infty} \frac{x^n}{n}$.)

   (a) $(-6, 0)$.

   (b) $(-1, 3)$.

   (c) Challenge: $[-1, 3)$.

Series Problems

1. Decide whether the following series converge or diverge. Explain your reasoning.

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{2n^5 + 500n^4 + n^3}{n^7 + 200n^6}$.

   (b) $\displaystyle\sum_{n=100}^{\infty} \frac{\sin n}{n^2}$.

2. Use a second degree Taylor polynomial to approximate $33^{1/5}$.

3. We define a function $f(x)$ by setting $f(x) = \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{\sqrt{n}2^n}$ for those x for which the series converges.

(a) Find the radius of convergence.

(b) Write a power series representation of $f'(x)$, the first derivative of f. Use it to find a series for $f'(1)$.

(c) Write out the first three non-zero terms of your series for $f'(1)$. At $x = 1$, is the function f increasing or decreasing? Explain.

Differential Equations: An Introduction to Modeling

In #1 - #8, write a differential equation that reflects the situation. Include an initial condition if the information is given.

1. The population of a certain country increases at a rate proportional to the population size. Let $P = P(t)$ be the population at time t.

2. A snowball melts at a rate proportional to its surface area. At time 0, the snowball has a radius of 10 cm. Let $r = r(t)$ be the radius of the snowball at time t.

3. A yellow rubber duck is dropped out of the window of an apartment building at a height of 80 feet. Let $s = s(t)$ be the height of the duck above the ground at time t. (Gravity is the acceleration -32 ft/s^2.)

4. Ferdinand is trying to fill a bucket from a faucet. Unfortunately, he doesn't realize that there is a small hole in the bottom of the bucket. Water flows in to the bucket from the faucet at a constant rate of .75 quarts per minute, and it flows out of the hole at a rate proportional to the amount of water $W(t)$ already in the bucket (due to the increased water pressure).

5. A drug is being administered to a patient at a constant rate of c mg/hr. The patient metabolizes and eliminates the drug at a rate proportional to the amount in his body. Let $M = M(t)$ be the amount (in mg) of medicine in the patient's body at time t, where t is measured in hours.

6. \$6000 is deposited in a bank account. The account has a nominal annual interest rate of 2%, compounded continuously. There are no deposits and no withdrawals. Let $M = M(t)$ be the amount of money in the account at time t, where t is measured in years.

7. \$6000 is deposited in a bank account. The account has a nominal annual interest rate of 2%, compounded continuously. Money is being withdrawn at a rate of \$500 per year.[1] Let $M = M(t)$ be the amount of money in the account at time t, where t is measured in years.

8. A rumor spreads at a rate proportional to product of the number of people who have heard it and the number who have not. In a town of N people, suppose 1 person originates the rumor at time $t = 0$. Let $y = y(t)$ be the number of people who have heard the rumor at time t.

   What does this model imply about the number of people who eventually have heard the rumor?

The following problems are about solutions to differential equations.

9. Which of the following is a solution to $\frac{dy}{dx} = y$?

   (a) $y = \frac{x^2}{2} + C$.

   (b) $y = e^x + C$.

   (c) $y = Ce^x$.

10. Give two solutions to $\frac{dy}{dx} = 5y$. What is the general solution?

11. Give two solutions to to $\frac{dy}{dx} = 5x$. What is the general solution?

[1] In reality, you cannot withdraw money continuously from the bank, but it's convenient to use a continuous model.

Slope Fields

1. Draw the slope fields for the following differential equations:

   (a) $\frac{dy}{dt} = 1$.

   (b) $\frac{dy}{dt} = t$.

   (c) $\frac{dy}{dt} = y$.

   (d) $\frac{dy}{dt} = \frac{-t}{y}$.

2. Draw the slope field for the differential equation $\frac{dy}{dt} = y - 1$. Sketch two solutions to the equation.

3. Which of the following is a solution to $\frac{dy}{dt} = y - 1$?

   (a) $y = Ce^t$

   (b) $y = Ce^t - t$

   (c) $y = Ce^{-t} - 1$

   (d) $y = Ce^t - 1$

   (e) $y = Ce^t + 1$

4. Which of the following is a solution to $y'' - y' - 6y = 0$?

   (a) $y = Ce^t$.

   (b) $y = C \sin 2t$.

   (c) $y = 5e^{3t} + e^{-2t}$.

   (d) $y = e^{3t} - 2$.

Separation of Variables / Mixing Problems

1. Find the general solution of the differential equation $\frac{dM}{dt} = 2.4 - .2M$. (Such a differential equation came up, for instance, when we modeled the amount of medicine in a patient's body.)

2. Last time, we solved the differential equation $\frac{dy}{dt} = -\frac{t}{y}$ by drawing the slope field, guessing the solution, and checking it. Now, solve the differential equation using separation of variables.

3. Solve the differential equation $\frac{dy}{dt} = e^{-t-y}$, and find the particular solution satisfying the initial condition $y(0) = 1$.

4. Solve the differential equation $y' = 2y - 6$.

5. Which of the following differential equations are separable? (You need not solve.)

   (a) $\frac{dy}{dt} = t + y$.

   (b) $\frac{dy}{dt} = \frac{y}{\sin t}$.

   (c) $\frac{dy}{dt} = \frac{\sin t}{y} + t$.

6. A 20-quart juice dispenser in a cafeteria is filled with a juice mixture that is 10% mango and 90% orange juice. A pineapple-mango blend (40% pineapple and 60% mango) is entering the dispenser at a rate of 4 quarts an hour and the well-stirred mixture leaves at a rate of 4 quarts an hour. Model the situation with a differential equation whose solution, $M(t)$, is the amount of mango juice in the container at time t. ($t = 0$ is the time when the pineapple-mango blend starts to enter the dispenser.)

7. Suppose that, in the previous problem, the mixture was leaving at a rate of 5 quarts per hour rather than 4 quarts per hour. Model the new situation.

Second-Order Homogeneous Differential Equations with Constant Coefficients

1. Suppose $f(t)$ and $g(t)$ are both solutions to the differential equation $y''+by'+cy=0$. Is $C_1f(t)+C_2g(t)$ a solution as well?

2. Can you guess solutions of $y''=y$? Try to guess two solutions that are not just multiples of each other.

3. Can you guess solutions of $y''=4y$? Try to guess two solutions that are not just multiples of each other.

4. Solve $y''-y'=6y$.

5. Solve $y'' + 5y' + 4y = 0$ where $y(0) = 1$ and $y'(0) = 2$.

6. Solve $y'' - 4y' + 4y = 0$.

7. Show that, if the characteristic equation $y'' + by' + cy = 0$ has one repeated root r, then $y = te^{rt}$ is a solution to $y'' + by' + cy = 0$.

1. Solve $y'' - 6y' + 9y = 0$.

2. Solve $y'' + y = 0$.

3. Solve $y'' - 2y' + 5y = 0$.

4. (a) Solve $y'' + 2y' + 4y = 0$ with initial conditions $y(0) = 1$ and $y'(0) = 0$.

   (b) Interpret part (a) in terms of a vibrating spring. What is happening to the spring as time goes on?

5. Which of the following differential equations has periodic solutions? What is the period?

   (a) $y'' + 2y' - 3y = 0$.

   (b) $y'' + 2y + 3y = 0$.

   (c) $y'' + 4y' = 0$.

   (d) $y'' + 4y = 0$.

   (e) $y'' - 4y = 0$.

   Does this agree with your interpretation of the differential equations in terms of vibrating springs?

1. Which of the following differential equations has periodic solutions? What is the period?

   (a) $y'' + 2y' - 3y = 0$.

   (b) $y'' + 2y' + 3y = 0$.

   (c) $y'' + 4y' = 0$.

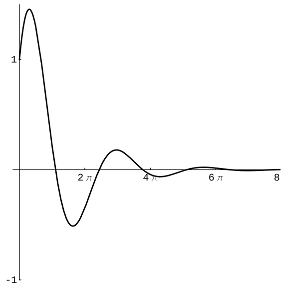
   (d) $y'' + 4y = 0$.

   (e) $y'' - 4y = 0$.

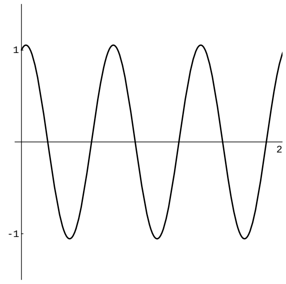
   Does this agree with your interpretation of the differential equations in terms of vibrating springs?

2. A spring with a mass of 5 kg has a natural length of 6 cm. A 20 N force is required to compress it to a length of 5 cm. If the spring is stretched to a length of 7 cm and released, find the position of the mass at time t. Sketch a graph of the position vs. time.

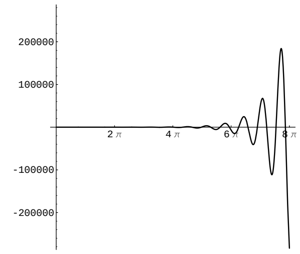
3. Match each differential equation with the graph of its solution. In each case, the differential equation has initial conditions $y(0) = 1$, $y'(0) = 1$.



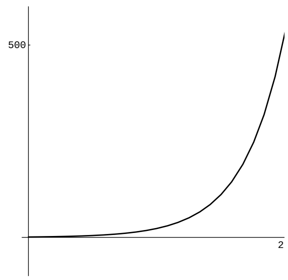
(1)



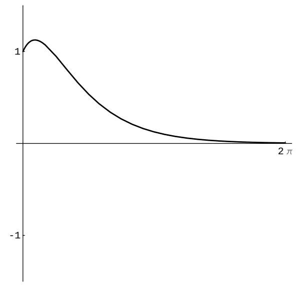
(2)



(3)



(4)



(5)

(a) $y'' + 3y' + 2y = 0.$

(b) $y'' + 9y = 0.$

(c) $y'' - 2y' + y = 0.$

(d) $y'' - y' + 10y = 0.$

(e) $y'' + \frac{1}{2}y' + \frac{5}{8}y = 0.$

4. Solve the differential equation $y'' + y = 0$ with initial conditions $y(0) = 0$ and $y'(0) = 1$.

Systems of Differential Equations

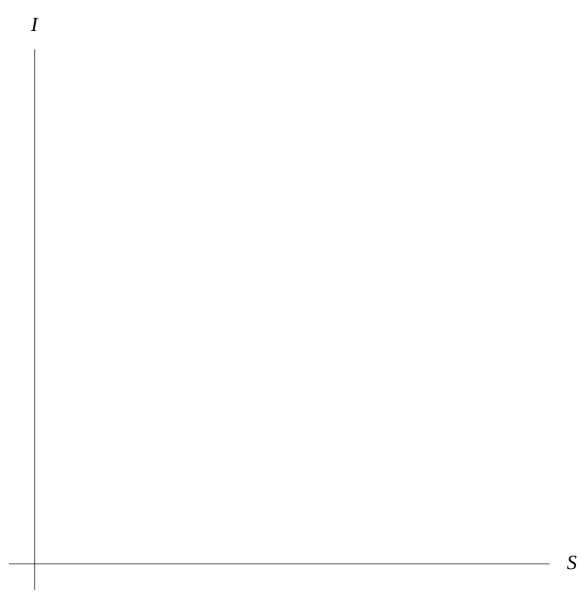
We've used systems of differential equations to model interaction between species. Systems can also be used to model disease epidemics.

Suppose that there is a large population of people, and some of the people have a fatal disease. This disease is infectious, so anybody who doesn't have the disease is susceptible to getting it. Let $I(t)$ be the number of people infected at time t, and let $S(t)$ be the number of people who are susceptible at time t.

1. How could you model this situation with a system of differential equations? You may ignore birth and death, except for death due to the disease, which you should include. (There are many many different answers; when in doubt, opt for simplicity.)

2. Using common sense, find the equilibrium points in this model. (You do not need to use the differential equations you found in #**??**; just think about the situation.)

3. Using common sense, sketch some typical phase trajectories in the phase plane.



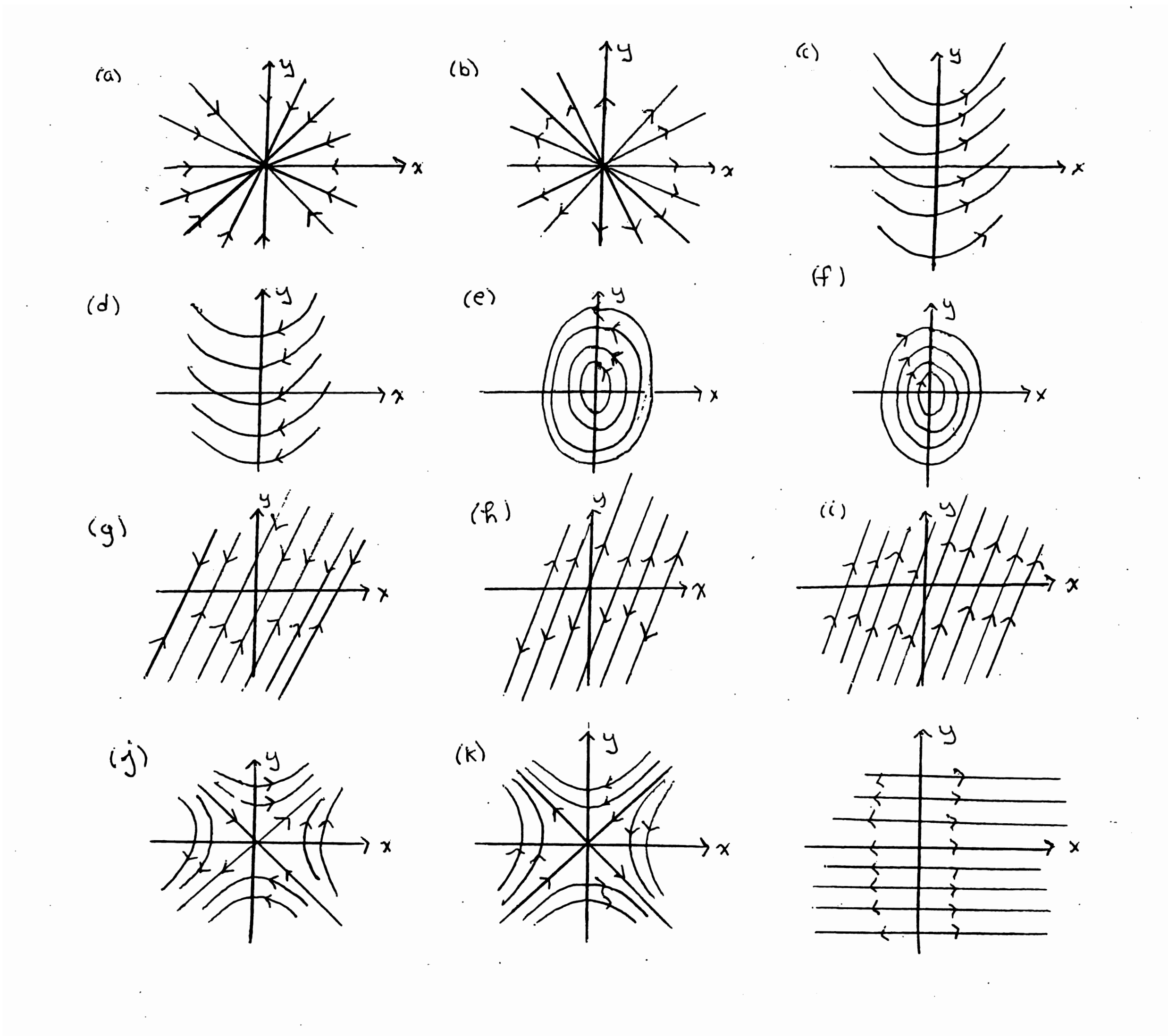
4. A reasonable system for the situation described is:

$$\begin{aligned}\frac{dS}{dt} &= -0.001IS \\ \frac{dI}{dt} &= 0.001IS - 0.1I\end{aligned}$$

Sketch the phase portrait for this system. (Be sure to draw the nullclines and equilibrium points.)

5. If the population starts with 50 infected people and 200 susceptible people, what will happen in the long run?

The problems refer to these diagrams:

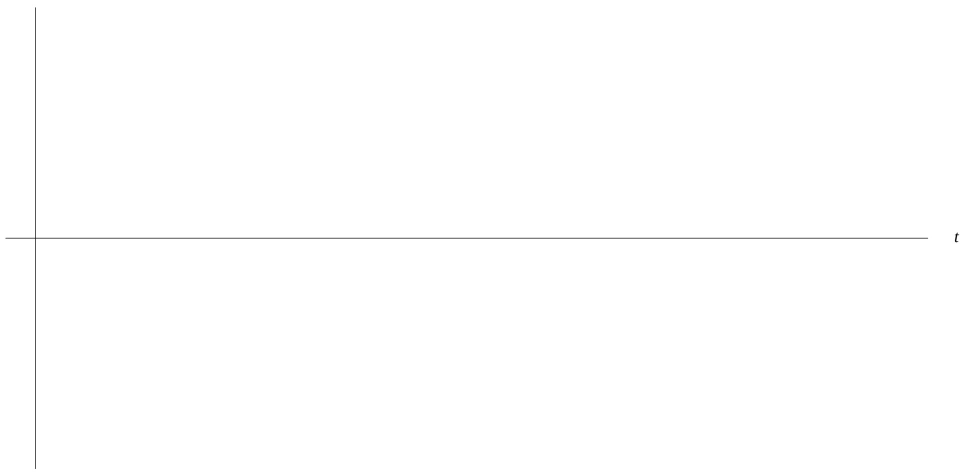


1. Find the diagram which matches the system.

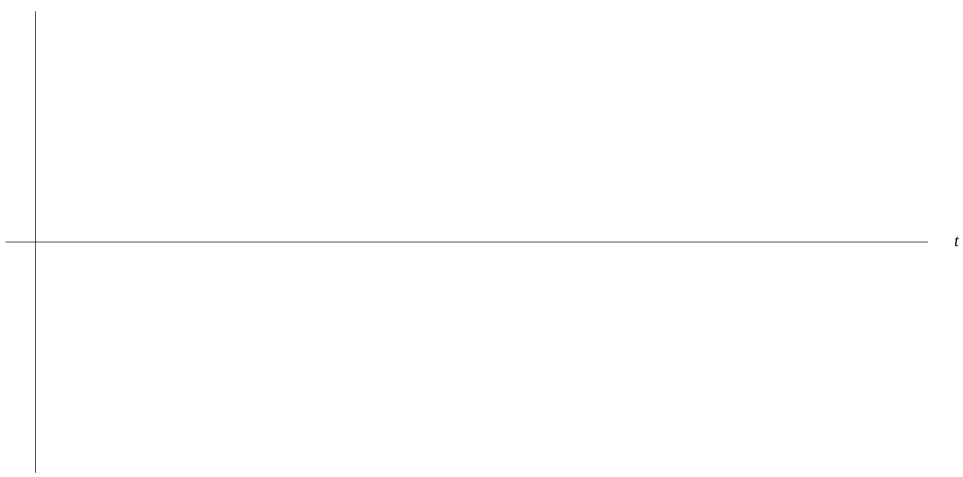
   (i) $\frac{dx}{dt} = y$ and $\frac{dy}{dt} = -2x$.

   (ii) $\frac{dx}{dt} = y$ and $\frac{dy}{dt} = 3y$.

2. (i) Suppose that the system $\frac{dx}{dt} = f(x, y)$ and $\frac{dy}{dt} = g(x, y)$ has phase portrait (e). Sketch possible graphs of $x(t)$ and $y(t)$, assuming $x(0) = 0$ and $y(0) = 3$.



   (ii) Do the same thing for (g).



3. Solve the system $\frac{dx}{dt} = y$ and $\frac{dy}{dt} = 3y$ with initial conditions $x(0) = 0$ and $y(0) = 3$.

4. Solve the system $\frac{dx}{dt} = y$ and $\frac{dy}{dt} = -2x$ with initial conditions $x(0) = 0$ and $y(0) = 3$.

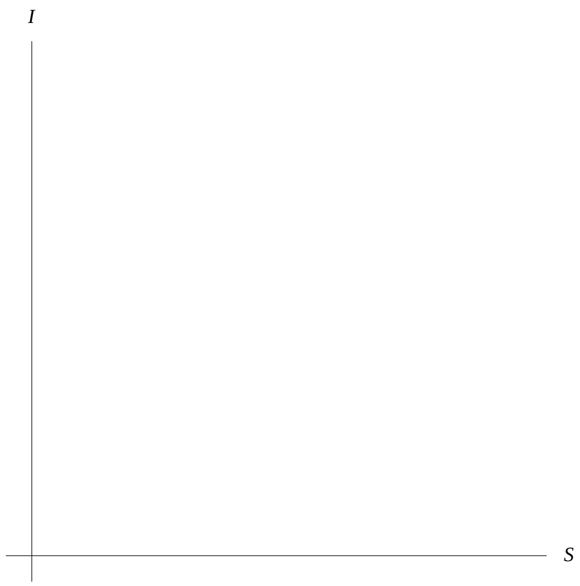
5. Find a system of differential equations whose phase portrait looks like (l), the last diagram.

Measles

We'll look at the system

$$\frac{dS}{dt} = -IS + 50$$
$$\frac{dI}{dt} = IS - 10I$$

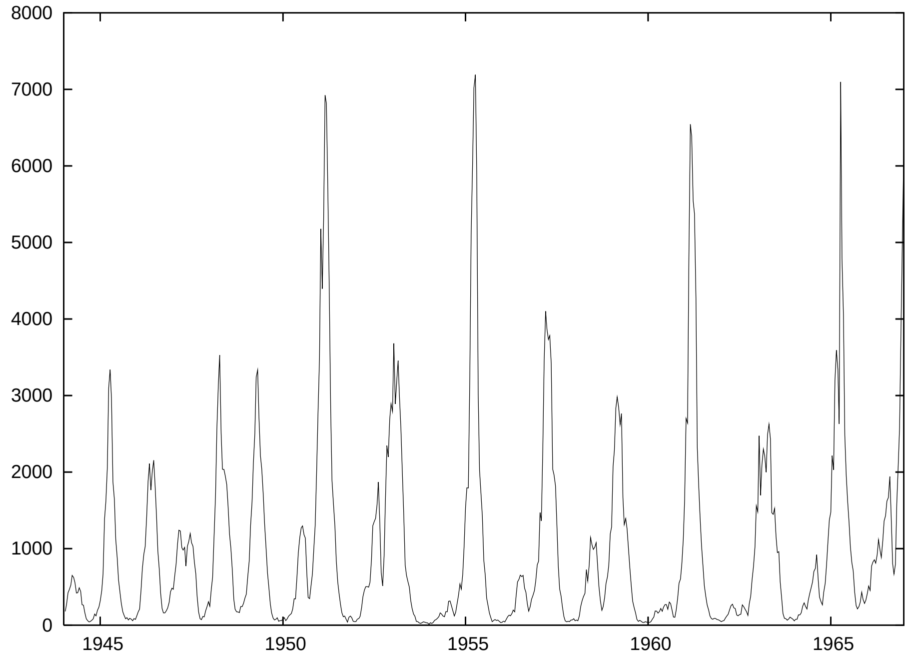
1. Do a qualitative phase plane analysis of this system. (You should draw equilibrium points, nullclines, and the direction of the trajectories in each region.)



2. Based on your phase plane analysis, what do you think the trajectories look like? Sketch a possible trajectory on your diagram if $S(0) = 5$ and $I(0) = 20$.

3. Using your trajectory, sketch a possible graph of $I(t)$ if $S(0) = 5$ and $I(0) = 20$.

4. This graph shows the number of cases of measles in 2 week periods in London from 1944 to 1966. Does our system give the same qualitative behavior?

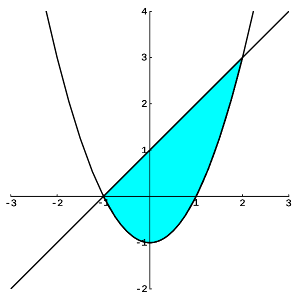


The Definite Integral

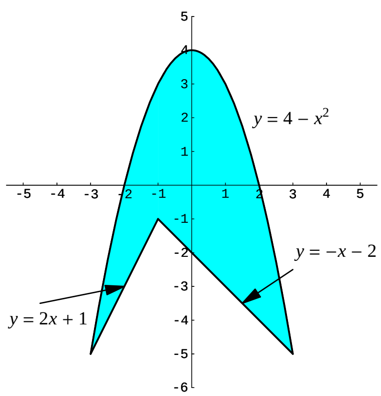
1. The definite integral is *defined* to be

   (a) a limit of Riemann sums.
   (b) the difference in the evaluation of an antiderivative at the endpoints of the interval.
   (c) a signed area.
   (d) all of the above.

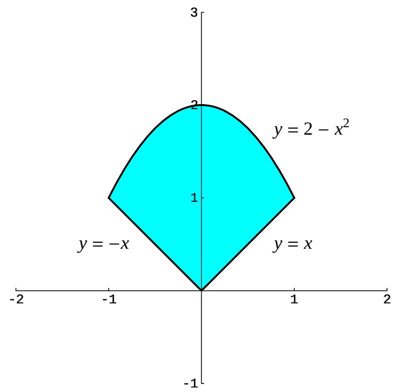
2. Write an integral or a sum and/or difference of integrals that gives the area enclosed by the graphs of $y = x^2 - 1$ and $y = x + 1$.



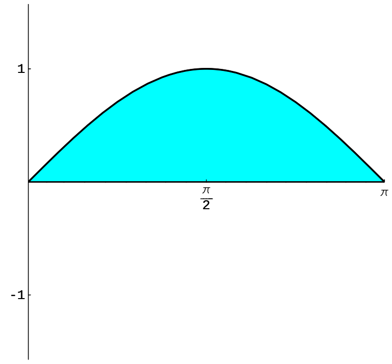
3. Find the area of the following region.



1. Find the area of the region shown. (You may leave your answer as an integral or sum/difference of integrals.)



2. Here is one loop of the sine curve.

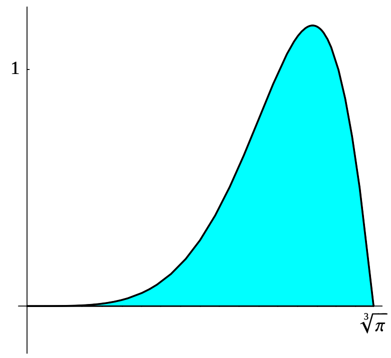


(a) If you rotate this region about the x-axis, what shape do you get? What is its volume? (You do not need to evaluate your integral.)

(b) If you rotate the region about the line $y = -1$, what shape do you get? What is its volume? (You do not need to evaluate your integral.)

More on Volumes

1. This is the curve $y = x \sin x^3$. If we rotate this region about the y-axis, what is the volume of the resulting solid? (Once you get an integral, try to evaluate it.)



2. How can you describe a bagel as a solid of revolution? (That is, what sort of region would you rotate, and what line would you rotate it about?)

3. The disk of radius 3 centered at the origin is rotated about the line $x = 4$. Find the volume using vertical slices. (It is also possible to do it using horizontal slices, and you might want to try that for extra practice.)

4. Let $\mathcal{R}$ be the region enclosed by the x-axis, the y-axis, $y = 1$, and $y = \frac{1}{x} - 1$.

   (a) Find the volume generated when $\mathcal{R}$ is rotated about the line $x = -2$.

   (b) Find the volume generated when $\mathcal{R}$ is rotated about the line $y = 2$.

Integration by Parts

Evaluate the following integrals.

1. $\int xe^x \; dx.$

2. $\int x \ln x \; dx.$

3. $\int_1^e \ln x \; dx.$

4. $\int_0^1 \arctan x \; dx.$

5. $\int x^2 \cos 2x \; dx.$

6. $\int e^x \cos x \; dx.$

7. $\int \cos \sqrt{x} \; dx.$

8. You are given the following information about an unknown function $g(x)$:

$$\int_1^2 \frac{g(u)}{u} \; du = 3, \int_1^2 g(u) \; du = 4, \int_1^4 g(u) \; du = 5, g(1) = 2, g(2) = -2.$$

(a) Evaluate $\int_1^2 (\ln x) g'(x) \; dx.$

(b) Evaluate $\int_1^2 x g(x^2) \; dx.$

9. $\int \sin 5x \sin 3x \; dx.$

Partial Fractions

1. Which of the following is easiest to integrate?

   (a) $\displaystyle\int \frac{5x-4}{x^2-x-2}\,dx.$

   (b) $\displaystyle\int \frac{5x-4}{(x-2)(x+1)}\,dx.$

   (c) $\displaystyle\int \frac{5x}{x^2-x-2}\,dx - \int \frac{4}{x^2-x-2}\,dx.$

   (d) $\displaystyle\int \frac{3}{x+1}\,dx + \int \frac{2}{x-2}\,dx$

   How do the four choices relate to each other?

2. Evaluate the following integrals.

   (a) $\displaystyle\int \frac{1}{y^2-4}\,dy.$

   (b) $\displaystyle\int \frac{5x-7}{x^2-3x+2}\,dx.$

3. Write down the form of the partial fraction expansion for the following integrals. (You don't need to actually solve for the coefficients.)

(a) $\displaystyle\int \frac{3x^2+x+5}{(x+1)(x+3)(x-5)}.$

(b) $\displaystyle\int \frac{x+1}{(x^2+4)(x^2+9)}.$

(c) $\displaystyle\int \frac{x^3+2x}{(x+4)(x+3)(x+2)^2}.$

(d) $\displaystyle\int \frac{x^2+1}{(x+1)^2(x^2+5)}.$

(e) $\displaystyle\int \frac{3x^2}{x^2+2x+1}\,dx.$

(f) $\displaystyle\int \frac{x^3+4x^2+7x}{x^2+4x+3}\,dx.$

4. Evaluate the following integrals.

(a) $\displaystyle\int \frac{x^2-x+4}{x^3+4x}.$

(b) $\displaystyle\int \frac{1}{x^3+x}\,dx.$

(c) $\displaystyle\int \frac{\sin\theta\;d\theta}{\cos^2\theta+\cos\theta-2}.$

Integration Techniques

In each problem, decide which method of integration you would use. If you would use substitution, what would u be? If you would use integration by parts, what would u and dv be? If you would use partial fractions, what would the partial fraction expansion look like? (Don't solve for the coefficients.)

1. $\displaystyle\int \frac{\cos x \; dx}{\sqrt{1+\sin x}}$.

2. $\displaystyle\int (\ln x)^2 \; dx$.

3. $\displaystyle\int e^x \sin x \; dx$.

4. $\displaystyle\int \frac{x}{x^2-1} \; dx$.

5. $\int xe^{x^2}\ dx.$

6. $\int \frac{x^2}{x^2+4x+3}\ dx.$

7. $\int \frac{e^t}{1+e^t}\ dt.$

8. $\int \arcsin x\ dx.$

Density and Slicing

1. A seaside village, Playa del Carmen, is in the shape of a rectangle 4 miles wide and 6 miles long. The sea lies along a 6-mile long side. People prefer to live near the water, so the density of people is given by $\rho(x) = 10000 - 800x$ people per square mile, where x is the distance from the seaside. We would like to find the population of the village.

   (a) Show in a sketch how to slice up the region.

   (b) What is the area of the k-th slice?

   (c) What is the approximate population in the k-th slice?

   (d) Write a general Riemann sum to estimate the total population of the city.

   (e) Find a definite integral expressing the population of the village.

2. A rectangular plot of farm land is 300 meters by 200 meters. A straight irrigation pipe 300 meters long runs down the center of the plot, dividing it in half lengthwise. The farmer's yield decreases as the distance from the irrigation pipe increases. Suppose that the yield is given by $\rho(x)$ grams per square meter, where x is the distance in meters from the irrigation pipe. Write an integral giving the total yield from the plot.

3. People in the Boston area like to live near the city center, so the population density around Boston is $\rho(r) = \frac{36{,}000}{r^2+2r+1}$ people per square mile, where r is the distance in miles to the center of Boston. Find the number of people who live within 5 miles of the center of Boston.

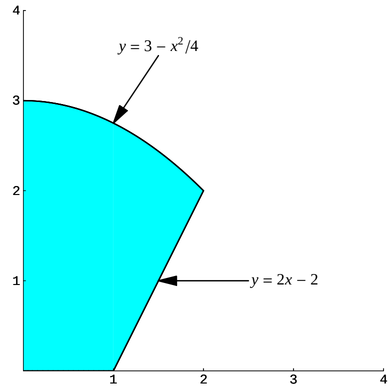
More Applications of Integration

1. A cone with height 8 inches and radius 6 inches is filled with flavored slush. When the cup is held upright with the pointed end resting on a table, the density of flavoring syrup in the cup varies with height above the table. Suppose $\rho(x)$ gives the number of ounces of syrup per cubic inch, where x is the distance from the table top. Write an integral giving the total amount of syrup in the cup.

2. Suppose the density of a planet is given by the function $\rho(r) = \dfrac{40000}{1 + 0.0001r^3}$ kilograms per cubic kilometer, where r is the distance in kilometers from the center of the planet. Find the total mass of the planet if its radius is 8000 km. (You do not need to evaluate your integral.)

3. A cylindrical candle of height 50 mm and radius 12 mm is formed by repeatedly dipping a wick of radius 1 mm into hot wax and then allowing the new layer of wax to dry. The density of each new layer is slightly different, so the density of the candle varies with the distance to the wick. If $\rho(x)$ gives the density in grams per cubic mm of the wax, where x measures the distance to the wick, write an integral giving the mass of the candle.

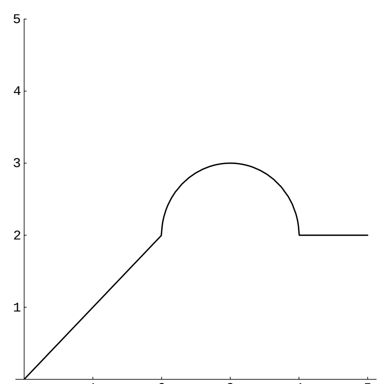
4. We can model a muffin as a solid of revolution, obtained by rotating the following region about the y-axis. Due to a poor recipe, the chocolate chips in our muffin tend to sink to the bottom. The amount of chocolate in the muffin is given by $\rho(y) = 5 - y$ grams per cubic inch, where y represents the distance to the bottom of the muffin. Find the total amount of chocolate in the muffin.



5. You ride your bike with velocity $v(t) = 3t^2 + 4t - 5$ in the time interval $[0, 3]$. What is your average velocity?

6. The temperature outside is given by the function $f(t)$, where t represents the time since 10:00 am. How would you find the average temperature between noon and 5:00 pm?

7. The graph of a function f is shown. The graph is made up of lines and semicircles. Find the average value of f on the interval $[1, 5]$.



Arc Length and Improper Integrals

1. *Write an integral that gives the length of one arch of the sine curve (so from $x = 0$ to $x = \pi$).*

   Solution. Our formula tells us that it is $\int_0^\pi \sqrt{1+\cos^2 x}\, dx$.

2. (a) *Does* $\int_1^\infty \frac{1}{x^2}\, dx$ *converge or diverge? If it converges, evaluate it.*

   Solution. We know that $\int_1^\infty \frac{1}{x^2}\, dx$ really means $\lim_{b\to\infty} \int_1^b \frac{1}{x^2}\, dx$. We can evaluate $\int_1^b \frac{1}{x^2}\, dx$ pretty easily: it is $-\frac{1}{x}\big|_1^b = -\frac{1}{b} + 1 = 1 - \frac{1}{b}$. So, $\int_1^\infty \frac{1}{x^2}\, dx = \lim_{b\to\infty} 1 - \frac{1}{b} = \boxed{1}$.

   (b) *Does* $\int_1^\infty \frac{1}{x}\, dx$ *converge or diverge? If it converges, evaluate it.*

   Solution. We know that $\int_1^\infty \frac{1}{x}\, dx$ really means $\lim_{b\to\infty} \int_1^b \frac{1}{x}\, dx$, so

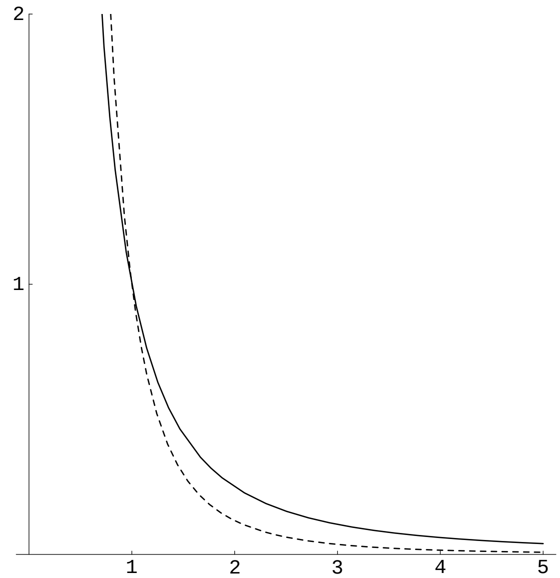
$$\begin{aligned} \int_1^\infty \frac{1}{x}\, dx &= \lim_{b\to\infty} \left(\ln|x| \big|_1^b \right) \\ &= \lim_{b\to\infty} (\ln|b| - \ln 1) \\ &= \lim_{b\to\infty} \ln|b| \text{ since } \ln 1 = 0 \end{aligned}$$

   But we know that $\lim_{b\to\infty} \ln|b| = \infty$, which is a form of diverging, so the improper integral $\int_1^\infty \frac{1}{x}\, dx$ diverges.

3. Using #2, can you conclude anything about whether the following integrals converge or diverge? (Try to figure this out without evaluating the integrals!)

   (a) $\int_1^\infty \frac{1}{x^3}\, dx$*?*

   Solution. Let's graph $\frac{1}{x^2}$ (the solid curve) and $\frac{1}{x^3}$ (the dashed curve):

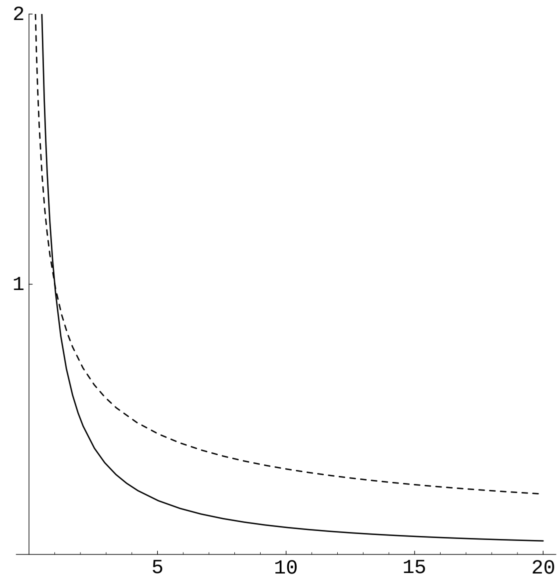


By 2(a), we know that $\int_1^\infty \frac{1}{x^2}\,dx = 1$. Graphically, we interpret this as the area under the curve $y = \frac{1}{x^2}$ to the right of $x = 1$. It is apparent from our picture that the area under $y = \frac{1}{x^3}$ to the right of $x = 1$ should be less than that, so we expect the integral $\int_1^\infty \frac{1}{x^3}\,dx$ to converge. In fact, it does:

$$\begin{aligned}\int_1^\infty \frac{1}{x^3}\,dx &= \lim_{b\to\infty}\int_1^b \frac{1}{x^3}\,dx\\ &= \lim_{b\to\infty}\left(-\frac{1}{2}x^{-2}\Big|_1^b\right)\\ &= \lim_{b\to\infty}\left(-\frac{1}{2b^2}+\frac{1}{2}\right)\\ &= \frac{1}{2}\end{aligned}$$

(b) $\int_1^\infty \frac{1}{x^{1/2}}\,dx$*?*

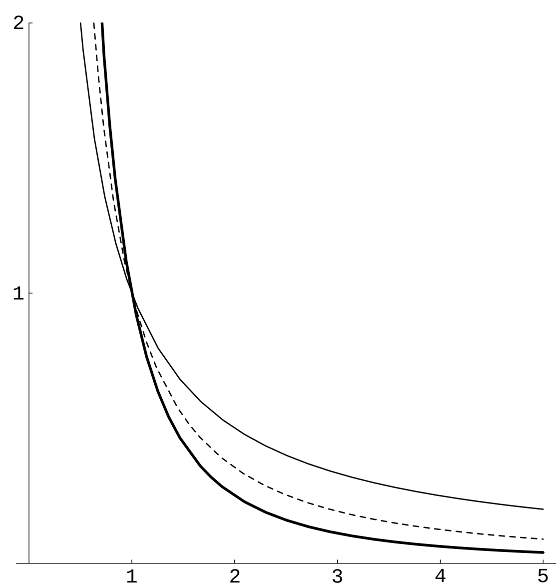
Solution. Let's graph $\frac{1}{x}$ (the solid curve) and $\frac{1}{x^{1/2}}$ (the dashed curve):



Since the graph of $\frac{1}{x^{1/2}}$ is higher than the graph of $\frac{1}{x}$ when $x \geq 1$, the area under $\frac{1}{x^{1/2}}$ to the right of $x = 1$ should be at least as big as the area under $\frac{1}{x}$ to the right of $x = 1$. The area under $\frac{1}{x}$ to the right of $x = 1$ was already infinite, so we expect $\displaystyle\int_1^\infty \frac{1}{x^{1/2}}\, dx$ to diverge.

(c) $\displaystyle\int_1^\infty \frac{1}{x^{3/2}}\, dx$?

Solution. The graph of $\frac{1}{x^{3/2}}$ lies between the graphs of $\frac{1}{x}$ and $\frac{1}{x^2}$:



Here, the thin solid graph is $y = \frac{1}{x}$, the thick solid graph is $y = \frac{1}{x^2}$, and the dashed graph is $y = \frac{1}{x^{3/2}}$. Thinking in terms of areas, we can guess that $\displaystyle\int_1^\infty \frac{1}{x^2}\, dx \leq \int_1^\infty \frac{1}{x^{3/2}}\, dx \leq \int_1^\infty \frac{1}{x}\, dx$.

Using our result from #2, this tells us that $1 \leq \int_1^\infty \frac{1}{x^{3/2}}\ dx \leq \infty$. Unfortunately, that doesn't give us enough information to determine whether $\int_1^\infty \frac{1}{x^{3/2}}\ dx$ converges.

In this case, comparing to $\frac{1}{x}$ and $\frac{1}{x^2}$ doesn't help, so we'll evaluate:

$$
\begin{aligned}
\int_1^\infty \frac{1}{x^{3/2}}\ dx &= \lim_{b\to\infty} \int_1^b \frac{1}{x^{3/2}}\ dx \\
&= \lim_{b\to\infty} \left(-2x^{-1/2} \Big|_1^b \right) \\
&= \lim_{b\to\infty} -2b^{-1/2} + 2 \\
&= \lim_{b\to\infty} 2 - \frac{2}{\sqrt{b}} \\
&= 2
\end{aligned}
$$

So, the improper integral converges to 2.

Improper Integrals

Determine whether the following integrals converge or diverge. Explain your reasoning.

1. $\displaystyle\int_{-2}^{2} \frac{x}{x^2-1}\, dx.$

   Solution. The integrand is discontinuous at $x = \pm 1$, so we know we need to split the integral. The improprieties are at -1 and 1, and each of our pieces should have at most one impropriety. So, let's split like this:

$$\int_{-2}^{2} \frac{x}{x^2-1}\, dx = \int_{-2}^{-1} \frac{x}{x^2-1}\, dx + \int_{-1}^{0} \frac{x}{x^2-1}\, dx + \int_{0}^{1} \frac{x}{x^2-1}\, dx + \int_{1}^{2} \frac{x}{x^2-1}\, dx.$$

   (You could choose a number other than 0 between -1 and 1.)

   Now, we have to evaluate each of the integrals on the right (and they are all improper). Let's first find an antiderivative of $\frac{x}{x^2-1}$ by substituting $u = x^2 - 1$:

$$\int \frac{x}{x^2-1}\, dx = \frac{1}{2}\int \frac{1}{u}\, du = \frac{1}{2}\ln|u| = \frac{1}{2}\ln|x^2-1|.$$

   Now, we start evaluating our four improper integrals using limits.

$$\begin{aligned}\int_{-2}^{-1} \frac{x}{x^2-1}\, dx &= \lim_{b\to -1^-} \int_{-2}^{b} \frac{x}{x^2-1}\, dx \\ &= \lim_{b\to -1^-} \frac{1}{2}\ln|x^2-1|\Big|_{-2}^{b} \\ &= \lim_{b\to -1^-} \left(\frac{1}{2}\ln|b^2-1| - \frac{1}{2}\ln 3\right)\end{aligned}$$

   As $b \to -1^-$, $b^2 - 1 \to 0$, so $\ln|b^2-1| \to -\infty$. Thus, this integral diverges.

   Since one of our four pieces diverges, we don't need to bother calculating the other pieces; we already know that the whole integral $\boxed{\text{diverges}}$.

2. $\displaystyle\int_{1}^{\infty} \frac{1}{x^4+2}\, dx.$

   Solution. The integrand here is very similar to $\frac{1}{x^4}$, and we know $\displaystyle\int_1^{\infty} \frac{1}{x^4}\, dx$ converges. This suggests that we use the Comparison Theorem.

   Notice that $0 \le \frac{1}{x^4+2} \le \frac{1}{x^4}$ for all x. Since $\displaystyle\int_1^{\infty} \frac{1}{x^4}\, dx$ converges, the Comparison Theorem tells us that $\displaystyle\int_1^{\infty} \frac{1}{x^4+2}\, dx$ also $\boxed{\text{converges}}$.

3. $\displaystyle\int_{0}^{\infty} \frac{1}{e^x+x}\, dx.$

Solution. Since we don't know how to find an antiderivative of $\frac{1}{e^x+x}$, we should use the Comparison Theorem. Since $0 \le \frac{1}{e^x+x} \le \frac{1}{e^x}$ for all x and you saw on your homework that $\int_0^\infty \frac{1}{e^x}\,dx$ converges, the Comparison Theorem tells us that $\int_0^\infty \frac{1}{e^x+x}\,dx$ also $\boxed{\text{converges}}$.

4. $\displaystyle\int_{-\infty}^{\infty} \sin x \; dx.$

Solution. We need to split up this integral because it has two improprieties: the $-\infty$ and the ∞. It doesn't really matter where we split it, so let's split it at 0: $\int_{-\infty}^{\infty} \sin x \; dx = \int_{-\infty}^{0} \sin x \; dx + \int_{0}^{\infty} \sin x \; dx$. Let's do $\int_0^\infty \sin x \; dx$ first. By definition, this is $\lim_{b\to\infty} \int_0^b \sin x \; dx = \lim_{b\to\infty} -\cos x|_0^b = \lim_{b\to\infty} (-\cos b + 1)$, which does not exist. Since this piece diverges, we know that the whole integral $\boxed{\text{diverges}}$.

5. $\displaystyle\int_1^\infty \frac{1+e^{-x}}{x} \; dx.$

Solution. We can use the Comparison Theorem: $\frac{1+e^{-x}}{x} \ge \frac{1}{x} \ge 0$ when $x \ge 1$. Since we know that $\int_1^\infty \frac{1}{x}\,dx$ diverges, $\int_1^\infty \frac{1+e^{-x}}{x}\,dx$ must $\boxed{\text{diverge}}$ as well.

6. $\displaystyle\int_1^\infty \frac{\cos^2 x}{x^2} \; dx.$

Solution. We can use the Comparison Theorem: $\frac{1}{x^2} \ge \frac{\cos^2 x}{x^2} \ge 0$. Since we know that $\int_1^\infty \frac{1}{x^2}\,dx$ converges, $\int_1^\infty \frac{\cos^2 x}{x^2}\,dx$ $\boxed{\text{converges}}$ as well.

Probability

Waiting times, shelf-lifes, and equipment failure times are often modeled by exponentially decreasing probability density functions.

1. Suppose $f(t) = 0$ for $t < 0$ and $f(t) = 0.5e^{-ct}$ for $t \geq 0$ is the probability density function for the lifetime of a particular toy (t in years).

   (a) For what value of c is this a probability density function?

   (b) What is the probability that the toy lasts over one year? (Is there any way to compute this without computing an improper integral?)

   (c) What is the median life of this type of toy?

2. A large number of students take an exam. 30% of the students receive a score of 70, 50% receive a score of 80, and 20% receive a score of 90. What is the average score on the exam?

3. The density function for the duration of telephone calls within a certain city is $p(x) = 0.4e^{-0.4x}$ where x denotes the duration in minutes of a randomly selected call.

   (a) What percentage of calls last one minute or less?

   (b) What percentage of calls last between one and two minutes?

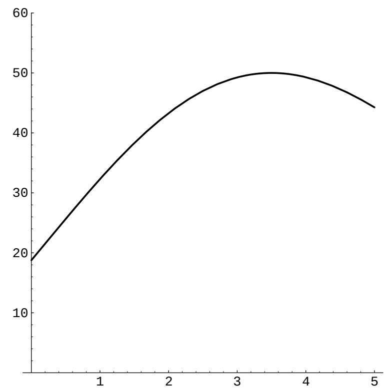
   (c) What percentage of calls last 3 minutes or more?

   (d) What is the average length of a call?

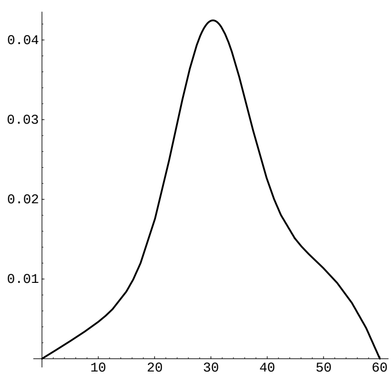
4. The lifetime, in hundreds of hours, of a certain type of light bulb has been found empirically to have a probability density function approximated by $f(x) = \frac{\sqrt{65}}{8(1+x^2)^{3/2}}$ for $0 < x < 8$. Find the mean lifetime of such a bulb.

More Probability

1. (a) The following function represents the temperature outside as a function of time. Estimate the average temperature between time 0 and time 5.



(b) A meteorologist takes several temperature readings which are described by the following probability density function. Estimate the average temperature.



2. Let $p(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$, a probability density function.

   (a) Sketch the graph of this probability density function. What do you think its mean is?

   (b) Verify your guess mathematically.

3. The bell curve with mean 0 and standard deviation s is given by the probability density function $p(x) = \frac{1}{s\sqrt{2\pi}} e^{-x^2/(2s^2)}$. What fraction of the population is within one standard deviation s of the mean 0?

Taylor Series

1. *Find the degree 6 Taylor polynomial approximation for $f(x) = \sin x$ centered at 0.*

   Solution. We are looking for a polynomial $P_6(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6$ such that the k-th derivative $P_6^{(k)}(0)$ is equal to the k-th derivative $f^{(k)}(0)$. The k-th derivative $P_6^{(k)}(0)$ is just equal to $k!a_k$, so we want $a_k = \frac{f^{(k)}(0)}{k!}$. The derivatives of $f(x) = \sin x$ are:

$$\begin{array}{rclcrcl} f(x) & = & \sin x & \Rightarrow & f(0) & = & 0 \\ f'(x) & = & \cos x & \Rightarrow & f'(0) & = & 1 \\ f''(x) & = & -\sin x & \Rightarrow & f''(0) & = & 0 \\ f'''(x) & = & -\cos x & \Rightarrow & f'''(0) & = & -1 \\ f^{(4)}(x) & = & \sin x & \Rightarrow & f^{(4)}(0) & = & 0 \\ f^{(5)}(x) & = & \cos x & \Rightarrow & f^{(5)}(0) & = & 1 \\ f^{(6)}(x) & = & -\sin x & \Rightarrow & f^{(6)}(0) & = & 0 \end{array}$$

   Therefore, $a_0 = 0$, $a_1 = 1$, $a_2 = 0$, $a_3 = -\frac{1}{3!}$, $a_4 = 0$, $a_5 = \frac{1}{5!}$, and $a_6 = 0$. So, $P_6(x) = \boxed{x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5}$.

2. (a) *If you want to find a Taylor polynomial approximation $a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$ (centered at 0) to $f(x)$, write a formula for the coefficient a_k.*

   Solution. $\boxed{a_k = \frac{f^{(k)}(0)}{k!}}$. (You might wonder what happens when $k = 0$: $0!$ is defined to be 1, so the formula still works.)

   (b) *If you want to find a Taylor polynomial approximation $a_0 + a_1(x-3) + a_2(x-3)^2 + \cdots + a_n(x-3)^n$ (centered at 3) to $f(x)$, write a formula for the coefficient a_k.*

   Solution. $\boxed{a_k = \frac{f^{(k)}(3)}{k!}.}$

3. *How do you think you would represent $\sin x$ as an infinite polynomial centered at 0? This is called the Taylor series (rather than Taylor polynomial) generated by $\sin x$ about 0.*

   Solution. Based on the pattern we started to see in #1, it seems like we should get

$$\boxed{x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \frac{1}{9!}x^9 - \cdots}.$$

4. *What is the Taylor series generated by $\cos x$ about 0?*

   Solution. Using $f(x) = \cos x$, we have

$$\begin{array}{rclcrcl} f(x) & = & \cos x & \Rightarrow & f(0) & = & 1 \\ f'(x) & = & -\sin x & \Rightarrow & f'(0) & = & 0 \\ f''(x) & = & -\cos x & \Rightarrow & f''(0) & = & -1 \\ f'''(x) & = & \sin x & \Rightarrow & f'''(0) & = & 0 \end{array}$$

After this, the derivatives repeat, and we continue to get $1, 0, -1, 0, 1, 0, -1, 0, \ldots$. So, our coefficients are $a_0 = 1$, $a_1 = 0$, $a_2 = -\frac{1}{2}!$, $a_3 = 0$, $a_4 = \frac{1}{4!}$, $a_5 = 0$, and so on. Thus, the Taylor series should be

$$\boxed{1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 - \cdots}.$$

5. *What is the Taylor series generated by e^x about 0?*

   Solution. If $f(x) = e^x$, then the k-th derivative $f^{(k)}(x)$ is always e^x, so $f^{(k)}(0) = 1$. Therefore, the k-th coefficient a_k in the Taylor series is $\frac{1}{k!}$. Thus, the Taylor series is $\boxed{1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots}$.

   If we wanted to write this in summation notation, we would write $\displaystyle\sum_{k=0}^{\infty} \frac{x^k}{k!}$.

6. We hope that, by using "polynomials of infinite degree," we end up with something that is not just an approximation for our function but is actually equal to the function. We don't really know if this is true yet. Taking on faith that e^x is actually equal to its Taylor expansion about 0, can you write a power series expansion (or "infinite polynomial representation") of:

   (a) e^{-x^2}*?*

   Solution. Since $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$, we can get e^{-x^2} just by replacing all of the x's in the series for e^x with $-x^2$: $e^{-x^2} = \boxed{1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \frac{x^8}{4!} - \cdots}$. In summation notation, $e^{-x^2} = \displaystyle\sum_{k=0}^{\infty} \frac{(-x^2)^k}{k!}$. We often simplify $(-x^2)^k$ as $[(-1)(x^2)]^k = (-1)^k x^{2k}$, so you might also see this as $\displaystyle\sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{k!}$.

   (b) $\displaystyle\int e^{-x^2}\, dx$*?*

   Solution. If we believe that $e^{-x^2} = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \frac{x^8}{4!} - \cdots$, then it seems plausible that we can integrate this using the reverse of the Power Rule to get $\displaystyle\int e^{-x^2}\, dx = C + x - \frac{1}{3}x^3 + \frac{1}{5} \cdot \frac{x^5}{2!} - \frac{1}{7} \cdot \frac{x^7}{3!} + \frac{1}{9} \cdot \frac{x^9}{4!} - \cdots$, where C is any constant.

7. (a) *Write a general formula for the Taylor series of $f(x)$ centered at 0.*

   Solution. We know it should look like $a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$ where $a_k = \frac{f^{(k)}(0)}{k!}$. So, it is $\boxed{f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots}$. In summation notation, it is $\displaystyle\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k$.

   (b) *What if you wanted to center at 5?*

   Solution. Then, we would get $a_0 + a_1(x-5) + a_2(x-5)^2 + a_3(x-5)^3 + \cdots$ where $a_k = \frac{f^{(k)}(5)}{k!}$. In other words, we would get $\boxed{f(5) + f'(5)(x-5) + \frac{f''(5)}{2!}(x-5)^2 + \frac{f'''(5)}{3!}(x-5)^3 + \cdots}$.

1. In your quest to become a millionaire by age 50, you start an aggressive savings plan. You open a new investment account on January 1, 2008 and deposit \$9000 into it every year on January 1. Each year, you earn 7% interest on December 31.

   (a) *How much money will you have in your account on January 2, 2009? 2010? 2014? (Don't try to add or multiply things out; just write an arithmetic expression.)*

   Solution. On December 31, 2008, you will have \$9000(1.07) because of the interest you've earned. After you deposit \$9000 on January 1, 2009, you will have $\$9000 + \$9000(1.07)$.

   On December 31, 2009, you will receive your interest, giving you $\$9000(1.07)+\$9000(1.07)^2$. After you make your yearly deposit on January 1, 2010, you will have $\$9000+\$9000(1.07)+\$9000(1.07)^2$.

   Continuing this reasoning, you will have $\$9000 + \$9000(1.07) + \$9000(1.07)^2 + \cdots + \$9000(1.07)^6$ on January 2, 2014.

   (b) *Will you be a millionaire by age 50?*

   Solution. The answer will depend on when you were born. Let's say that you were born in 1988. Then, we want to know what has happened by the year 2038. Using the argument of part (a), on January 2, 2038, you will have $\$9000 + \$9000(1.07) + \$9000(1.07)^2 + \cdots + \$9000(1.07)^{30}$. To see whether this is bigger than a million, we want some way to calculate this sum quickly.

   Let's call this amount S:

$$S = 9000 + 9000(1.07) + 9000(1.07)^2 + 9000(1.07)^3 + \cdots + 9000(1.07)^{29} + 9000(1.07)^{30} \qquad (1)$$

   Notice that if we multiply both sides by 1.07, we get something similar looking on the right side:

$$1.07S = 9000(1.07) + 9000(1.07)^2 + 9000(1.07)^3 + \cdots + 9000(1.07)^{30} + 9000(1.07)^{31} \qquad (2)$$

   Subtracting (**??**) from (**??**), most of the terms on the right side cancel, and we are left with $0.07S = 9000(1.07)^{31} - 9000$, so $S = \frac{9000(1.07)^{31}-9000}{.07} = \$918,657.37$. So, you are not quite a millionaire, but you are close!

2. If you suffer from allergies, your doctor may suggest that you take Claritin once a day. Each Claritin tablet contains 10 mg of loratadine (the active ingredient). Every 24 hours, about 7/8 of the loratadine in the body is eliminated (so 1/8 remains). [1]

   (a) *If you take one Claritin tablet every morning for a week, how much loratadine is in your body right after you take the 3rd tablet? 7th tablet? (Don't try to simplify your computations; just write out an arithmetic expression.)*

   Solution. Immediately after taking the first tablet, you have 10 mg of loratadine in your body. The following morning, only 1/8 of that is left, so you have 10 (1/8) mg in your body. You then take another pill containing 10 mg, so you have a total of $10 + 10(1/8)$ mg of loratadine in your body.

   The following morning, 1/8 of that remains, or $[10 + 10(1/8)](1/8) = 10(1/8) + 10(1/8)^2$. You then take another pill containing 10 mg, so you have a total of $10 + 10(1/8) + 10(1/8)^2$ mg of loratadine in your body after taking the 3rd pill.

[1] This estimate comes from the fact that the average half-life of loratadine is known to be 8 hours.

Continuing this reasoning, you will have $10+10(1/8)+10(1/8)^2+10(1/8)^3+10(1/8)^4+10(1/8)^5+10(1/8)^6$ mg in your body after the 7th pill.

(b) *If you take Claritin for years and years, will the amount of loratadine in your body level off? Or will your bloodstream be pure loratadine?*

Solution. Right after you take the n-th pill, you will have $10 + 10(1/8) + 10(1/8)^2 + 10(1/8)^3 + \cdots + 10(1/8)^{n-1}$ mg of loratadine in your body. Let's call this amount S_n. We are wondering what happens to S_n as n gets very large.

We use the same trick we used in #1(b) to get a closed form expression for S_n. We said that

$$S_n = 10 + 10(1/8) + 10(1/8)^2 + \cdots + 10(1/8)^{n-2} + 10(1/8)^{n-1} \quad (3)$$

Multiplying both sides by 1/8, we get that

$$(1/8)S_n = 10(1/8) + 10(1/8)^2 + 10(1/8)^3 + \cdots + 10(1/8)^{n-1} + 10(1/8)^n \quad (4)$$

If we subtract (**??**) from (**??**), most of the terms on the right side cancel, and we are left with $(7/8)S_n = 10 - 10(1/8)^n$. Dividing both sides by 7/8, $S_n = \frac{10-10(1/8)^n}{7/8}$.

Using this expression for S_n, it is easy to see what happens as n gets bigger and bigger: $(1/8)^n$ gets closer and closer to 0, so $\frac{10-10(1/8)^n}{7/8}$ gets closer and closer to $\frac{10}{7/8} = \frac{80}{7}$. Thus, over time, the amount of loratadine in your body gets closer and closer to $\frac{80}{7}$ mg.

3. *For what values of x does the geometric series $1 + x + x^2 + \cdots$ converge?* [2] *If it converges, what does it converge to?*

Solution. This is a geometric series $a + ar + ar^2 + ar^3 + \cdots$ with $a = 1$ and $r = x$. Therefore, we know that it diverges when $|x| \geq 1$. When $|x| < 1$, it converges to $\frac{1}{1-x}$.

4. Which of the following series are geometric?

(a) $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k 2^{k+1}}{3^k}$.

Solution. A geometric series is a series of the form $a + ar + ar^2 + \cdots$. To decide whether the given series is geometric, we want to see if it matches this form. It's helpful to write out the first few terms. When $k = 1$, we have $\frac{(-1)^1 2^2}{3^1} = -\frac{4}{3}$. When $k = 2$, we have $\frac{(-1)^2 2^3}{3^2} = \frac{8}{9}$. When $k = 3$, we have $\frac{(-1)^3 2^4}{3^3} = -\frac{16}{27}$. When $k = 4$, we have $\frac{(-1)^4 2^5}{3^4} = \frac{32}{81}$. So far, it looks like it could be the geometric series $a + ar + ar^2 + ar^3 + \cdots$ with $a = -\frac{4}{3}$ and $r = -\frac{2}{3}$.

To see if this is correct, let's compare the k-th terms. The k-th term of the given series is $\frac{(-1)^k 2^{k+1}}{3^k}$, while the k-th term of the geometric series is ar^{k-1}. So, we are hoping that $\frac{(-1)^k 2^{k+1}}{3^k} = \left(-\frac{4}{3}\right)\left(-\frac{2}{3}\right)^{k-1}$. If you multiply out the right side, you will see that this is indeed the case, so the given series is geometric with $a = -\frac{4}{3}$ and $r = -\frac{2}{3}$.

(b) $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^3}$.

[2] We could also write this series in summation notation as $\displaystyle\sum_{k=0}^{\infty} x^k$.

Solution. If we write this series out, it is $1 + \frac{1}{8} + \frac{1}{27} + \frac{1}{64} + \cdots$. We can already see that it's not geometric because the terms don't have a common ratio. (To elaborate: if it was geometric, the first term would be a and the second term would be ar; this means that a would have to be 1, and r would have to be $\frac{1}{8}$. But then the third term isn't right.)

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{2}{3^{n/2}}$.

Solution. We could also write this series as $\frac{2}{3^{1/2}} + \frac{2}{3} + \frac{2}{3^{3/2}} + \frac{2}{3^2} + \cdots$, which looks like it might be geometric with $a = \frac{2}{3^{1/2}}$ and $r = \frac{1}{3^{1/2}}$.

To check if this is correct, we want to see if the n-th term is $ar^{n-1} = \frac{1}{3^{1/2}} \left(\frac{1}{3^{1/2}}\right)^{n-1}$, and it is. So, the series is geometric with $a = \frac{2}{3^{1/2}}$ and $r = \frac{1}{3^{1/2}}$.

Let $a_1 + a_2 + a_3 + \cdots = \sum_{k=1}^{\infty} a_k$ be an infinite series, and let s_n denote its n-th partial sum: $s_n = a_1 + a_2 + a_3 + \cdots + a_n$.

- If $\lim_{n\to\infty} s_n = s$ for a finite number s, we say the series $a_1 + a_2 + a_3 + \cdots$ converges (to s) and write $a_1 + a_2 + a_3 + \cdots = s$.
- If $\lim_{n\to\infty} s_n$ does not exist (or is not finite), we say the series $a_1 + a_2 + a_3 + \cdots$ diverges.

1. Suppose you know that the infinite series $a_1 + a_2 + a_3 + \cdots + a_n + \cdots$ converges to s and that $a_k > 0$ for k any positive integer. Let $s_n = a_1 + a_2 + a_3 + \cdots + a_n$. For each of the following statements, determine whether the statement must be true, could possibly be true, or must be false.

   (a) $\lim_{n\to\infty} a_n = 0$.

   (b) $\lim_{n\to\infty} s_n = 0$.

   (c) There exists a number M such that $s_n < M$ for all n. (This is equivalent to saying that the partial sums are bounded. Why?)

   (d) $\sum_{k=5}^{\infty} a_k$ converges.

2. Suppose you know that $\lim_{n\to\infty} b_n = 0$. Can you be sure that the infinite series $b_1 + b_2 + b_3 + \cdots$ converges?

A sequence is an ordered list of numbers. Some examples are:

- $-2, -4, -6, -8, \ldots$
- $-\frac{1}{2}, \frac{1}{4}, -\frac{1}{8}, \frac{1}{16}, \ldots$

Sometimes sequences are also written in the form $\{s_n\}$ where s_n is the n-th term of the sequence. For instance, the two examples above are $\{-2n\}$ and $\left\{\left(-\frac{1}{2}\right)^n\right\}$.

3. (a) Give an example of a sequence (ordered list) of numbers such that the numbers are increasing but are bounded.

   (b) Give an example of a sequence (ordered list) of numbers such that the numbers are increasing and are not bounded.

   (c) Give an example of a sequence (ordered list) of numbers such that the numbers are bounded but have no limit as $n \to \infty$.

A sequence is monotonic if it is either increasing (each term is greater than or equal to the preceding term) or decreasing (each term is less than or equal to the preceding term).

4. (a) A sequence which is both monotonic and bounded

   must converge   could either converge or diverge   must diverge

   (b) A sequence which is monotonic but not bounded

   must converge   could either converge or diverge   must diverge

5. Consider the series $\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$ (called the harmonic series).

   (a) Does the sequence of terms converge? If so, to what does it converge?

   (b) Does the sequence of partial sums converge? If so, to what does it converge?

   (c) Does the series $\sum_{k=1}^{\infty} \frac{1}{k}$ converge?

   (d) Does the series $\sum_{k=10^{10}}^{\infty} \frac{1}{100000000k}$ converge?

6. See if you can determine whether each of the following series converges or diverges by using the Nth Term Test for Divergence, results about geometric series, or some sort of comparison to series you know about.

   (a) $\sum_{k=100}^{\infty} \frac{1}{3k}$.

   (b) $\sum_{k=3}^{\infty} \frac{(-1)^k 2^k}{3^k}$.

   (c) $\sum_{k=4}^{\infty} \frac{(-1)^k 3^k}{2^k}$.

   (d) $\sum_{n=1}^{\infty} \frac{\ln n}{n}$.

   (e) $\sum_{n=2}^{\infty} \frac{n}{\ln n}$.

   (f) $\sum_{n=0}^{\infty} \sin n$.

Comparison

1. Use the Comparison Test (also known as "direct comparison") to decide whether the following series converge or diverge.

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}3^n}$.

   (b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n!}$.

   (c) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{n^3 - n + 1000}$.

   (d) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\ln(1+n)}$.

2. True or false: If $\{a_n\}$ is a sequence with positive terms and $\lim_{n\to\infty} a_n = 0$, then there is a number k such that $a_n < 1$ whenever $n \geq k$.

3. Decide whether the following series converge or diverge using any method you like.

(a) $\displaystyle\sum_{n=100}^{\infty} \cos n.$

(b) $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k 2^{k+1}}{3^k}.$

(c) $1 + 0 + (-1) + 1 + 0 + (-1) + 1 + 0 + (-1) + \cdots.$

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n}.$

More Comparison

1. For what values of p is the <u>p-series</u> $\sum_{n=1}^{\infty} \frac{1}{n^p}$ convergent?

2. Does the series $\sum_{n=2}^{\infty} \frac{1}{n^2 - 1}$ converge or diverge?

3. Does the series $\sum_{k=1}^{\infty} \frac{3}{8^k - 2}$ converge or diverge?

4. Otto is given the following problem for homework.

> Decide whether the series $\sum_{n=1}^{\infty} \sin^2(\pi n)$ converges or diverges. Explain your reasoning.

Otto writes

> The improper integral $\int_1^{\infty} \sin^2(\pi x)\, dx$ diverges, so $\sum_{n=1}^{\infty} \sin^2(\pi n)$ also diverges by the Integral Test.

Otto is correct that the improper integral diverges (although he should have shown more work!). But the rest of his reasoning is incorrect — why? And what is the correct answer to the problem?

5. Let $\sum a_n$ and $\sum b_n$ be series with positive terms. The Limit Comparison Test only applies when $\lim_{n\to\infty} \frac{a_n}{b_n}$ is a positive real number.

(a) If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$, can you draw any conclusions?

(b) If $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$, can you draw any conclusions?

1. *Does the series* $\displaystyle\sum_{k=1}^{\infty}(-1)^{k+1}\frac{1}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots$ *converge or diverge? (This series is often called the alternating harmonic series.)*

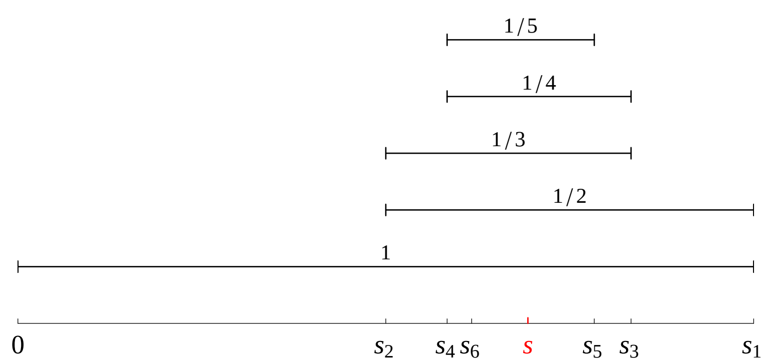
   Solution. To see if the series converges, let's see if we can apply the Alternating Series Test. The terms are decreasing in magnitude: $\frac{1}{k+1} \le \frac{1}{k}$. In addition, the terms approach 0: $\displaystyle\lim_{k\to\infty}\frac{1}{k} = 0$. Therefore, the Alternating Series Test applies, and we can conclude that $\displaystyle\sum_{k=1}^{\infty}(-1)^{k+1}\frac{1}{k}$ converges.

2. *In fact,* $1 - \dfrac{1}{2} + \dfrac{1}{3} - \dfrac{1}{4} + \dfrac{1}{5} - \cdots = \ln 2$. *Write a finite sum which estimates* $\ln 2$ *with error of less than* 0.001. *Is your approximation too big or too small?*

   Solution. The Alternating Series Estimate tells us that the magnitude of the error in using the n-th partial sum $s_n = \displaystyle\sum_{k=1}^{n}(-1)^{k+1}\frac{1}{k}$ is at most the magnitude of the next term, or $\frac{1}{n+1}$. That is, $|s - s_n| \le \frac{1}{n+1}$. We want $|s - s_n| < 0.001$, so let's find an n satisfying $\frac{1}{n+1} < 0.001$. (Then we'll have $|s - s_n| \le \frac{1}{n+1} < 0.001$.)

   To get $\frac{1}{n+1} < 0.001$, we need $n + 1 > 1000$. The smallest n which makes this work is $n = 1000$, so we can use the 1000th partial sum $\boxed{1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots - \frac{1}{999} + \frac{1}{1000}}$.

   To determine whether our approximation is too big or too small, let's go back and look at our diagram of partial sums. Here it is, with the actual sum s in red:



   From the diagram, we can see that the even partial sums $s_2, s_4, s_6, \ldots$ are all $\boxed{\text{too small}}$.

3. *Is the alternating harmonic series* $\displaystyle\sum_{k=1}^{\infty}(-1)^{k+1}\frac{1}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots$ *absolutely convergent?*

Solution. Asking whether the series is absolutely convergent is the same as asking whether the series $\sum_{n=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots$ converges. This is the harmonic series, which diverges. Thus, the series $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k}$ is not absolutely convergent.

4. Determine whether each series converges or diverges. If it converges, does it converge absolutely or conditionally?

   (a) $\sum_{k=1}^{\infty} \frac{(-1)^k}{\sqrt{k}}$.

   Solution. We can apply the Alternating Series Test: $\frac{1}{\sqrt{k+1}} \le \frac{1}{\sqrt{k}}$ and $\lim_{k\to\infty} \frac{1}{\sqrt{k}} = 0$, so the Alternating Series Test says that $\sum_{k=1}^{\infty} \frac{(-1)^k}{\sqrt{k}}$ converges.

   To decide whether it converges absolutely, we look at the series of absolute values, $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$. This is a p-series with $p = \frac{1}{2}$, and we know that diverges. So, $\sum_{k=1}^{\infty} \frac{(-1)^k}{\sqrt{k}}$ $\boxed{\text{converges conditionally}}$.

   (b) $\sum_{k=1}^{\infty} \frac{\sin k}{k!}$.

   Solution. This series is not an alternating series, so we should not even try the Alternating Series Test. All of our other tests involve series with positive terms, so let's look at the series of absolute values first. That is, we'll first look at $\sum_{k=1}^{\infty} \frac{|\sin k|}{k!}$.

   We can use the Comparison Test here: $0 \le \frac{|\sin k|}{k!} \le \frac{1}{k!}$. We know that $\sum_{k=1}^{\infty} \frac{1}{k!}$ converges (see #1(b) from the "Comparison" handout), so $\sum_{k=1}^{\infty} \frac{|\sin k|}{k!}$ converges.

   This means that $\sum_{k=1}^{\infty} \frac{\sin k}{k!}$ $\boxed{\text{converges absolutely}}$. We know that a series which converges absolutely also converges.

   (c) $\sum_{k=1}^{\infty} (-2)^k$.

   Solution. This is an alternating series, but the Alternating Series Test does not apply because the terms are not decreasing in magnitude. That does tell us whether the series converges though.

   Instead, notice that the terms aren't going to 0: $\lim_{k\to\infty} (-2)^k$ does not exist. So, the Nth Term Test for Divergence says that the series $\boxed{\text{diverges}}$. (Alternatively, you could justify by saying that the series is geometric with common ratio -2.)

5. (a) *The Taylor series for* $\cos x$ *about 0 is* $1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{k=0}^{\infty}(-1)^{k+1}\frac{x^{2k}}{(2k)!}$. *Show that, if you plug in any value of* x *with* $-0.5 \leq x \leq 0.5$, *the series converges.*

Solution. We will show that we can apply the Alternating Series Test. If $-0.5 \leq x \leq 0.5$, then $\frac{x^{2(k+1)}}{[2(k+1)]!} \leq \frac{x^{2k}}{(2k)!}$. This is true because the expression on the left has a bigger denominator and smaller numerator than the expression on the right (in the numerator, x is raised to a higher exponent, and $|x| < 1$). So, the first condition in the Alternating Series Test is satisfied.

Next, we need to show that $\lim_{k\to\infty} \frac{x^{2k}}{(2k)!} = 0$. In fact, $\lim_{k\to\infty} x^{2k} = 0$ (since x is between -0.5 and 0.5), so $\lim_{k\to\infty} \frac{x^{2k}}{(2k)!} = 0$ as well.

Therefore, the Alternating Series Test applies, and we can conclude that the series converges.

(b) *In fact, the series converges for all* x, *and* $\cos x$ *is actually equal to the series; that is,*

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots.$$

Suppose you use the approximation $\cos x \approx 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$ *to approximate* $\cos x$ *when* $-0.5 \leq x \leq 0.5$. *Find an upper bound for the error. (This means: find a number* U *that you can show is bigger than the error.)*

Solution. The Alternating Series Estimation Theorem tells us that the magnitude of the error is at most that of the first unused term, which is $\frac{x^6}{6!}$. Since $|x| \leq 0.5$, the error is at most $\frac{0.5^6}{6!} \approx 0.0000217014$.

Ratio Test

1. What does the Ratio Test tell you about the following series?

   (a) $\displaystyle\sum_{k=0}^{\infty}(-1)^{k+1}\frac{1000^k}{k!}$.

   (b) $\displaystyle\sum_{k=1}^{\infty}\frac{1}{k}$.

   (c) $\displaystyle\sum_{k=1}^{\infty}\frac{1}{k^2}$.

2. When we studied Taylor series, we found that the Taylor series for $\sin x$ about 0 was $x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$, which can be written in summation notation as $\displaystyle\sum_{k=0}^{\infty}(-1)^k\frac{x^{2k+1}}{(2k+1)!}$. For what values of x does this series converge?

3. When we studied Taylor series, we found that the Taylor series for $\ln(1+x)$ about 0 was $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$, which can be written in summation notation as $\displaystyle\sum_{k=1}^{\infty}(-1)^{k+1}\frac{x^k}{k}$. For what values of x does this series converge?

4. Decide whether the following series converge absolutely, converge conditionally, or diverge. You may use any method you like, but explain your reasoning. There is one that you will not be able to do (this is not due to a personal failing; it's just that all of the tests that we know are inconclusive).

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{\cos n}{n^2}$.

(b) $\displaystyle\sum_{n=100}^{\infty} \frac{n!n!}{(2n)!}$.

(c) $\displaystyle\sum_{n=0}^{\infty} \frac{\sin n}{n}$.

(d) $\displaystyle\sum_{n=2}^{\infty} \frac{\ln n}{n}$.

(e) $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{n^3+1}$.

(f) $\displaystyle\sum_{n=5000}^{\infty} (-1)^{n} \frac{n}{n+1}$.

Power Series

A power series centered at the number a is a series of the form $\sum_{n=0}^{\infty} c_n(x-a)^n$ where x is a variable and the c_n are constants.

1. For what values of x does the power series $\sum_{n=1}^{\infty} n!x^n$ converge? (This series is centered at 0.)

Theorem. For a given power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ centered at a, there are 3 possibilities:

1. The series converges only when $x = a$.
2. The series converges for all x.
3. There is a positive number R such that the series converges when $|x-a| < R$ and diverges when $|x-a| > R$. R is called the radius of convergence. (Note that this doesn't say anything about what happens when $|x-a| = R$.)

The interval of convergence of a power series is the set of x for which the power series converges.

2. Find the radius of convergence and interval of convergence of the power series $\sum_{n=1}^{\infty} \frac{2^n}{n}(x-3)^n$.

3. We know that the power series $\sum_{n=0}^{\infty} x^n$ converges to $\frac{1}{1-x}$ when $|x| < 1$. Find a power series representation of the function $\frac{x}{1+4x^2}$. What is the radius of convergence of this power series?

Theorem. If the power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ has radius of convergence R where $R > 0$ or $R = \infty$, then the function $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$ is differentiable on the interval $(a-R, a+R)$ and

1. $f'(x) = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1}$.

2. $\int f(x)\ dx = C + \sum_{n=0}^{\infty} \frac{c_n}{n+1}(x-a)^{n+1}$.

The power series in (1) and (2) both have radius of convergence R. (Note: Although the radius of convergence remains unchanged, the interval of convergence may change.)

4. (a) Find a power series representation for $\ln(1+x)$ centered at 0. What is the radius of convergence for the power series you have found? (Hint: What is the derivative of $\ln(1+x)$?)

   (b) Find the degree 5 Taylor polynomial approximation of $\ln(1+x)$.

5. Find a power series representation of $\arctan(5x)$ centered at 0. What is the radius of convergence of the power series you have found?

More on Power Series

1. Suppose we have a power series $\sum_{n=1}^{\infty} c_n(x+7)^n$.

   (a) If you know that the power series converges when $x = 0$, what conclusions can you draw?

   (b) Suppose you also know that the power series diverges when $x = 1$. Now what conclusions can you draw?

   (c) Does $\sum_{n=1}^{\infty} c_n$ converge (assuming that the power series converges when $x = 0$ and diverges when $x = 1$)?

   (d) Does $\sum_{n=1}^{\infty} c_n(-8.1)^n$ converge?

   (e) Does $\sum_{n=1}^{\infty} c_n(-8)^n$ converge?

Theorem. If $f(x)$ has a power series representation centered at a (that is, $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$ for $|x-a| < R$), then that power series must be the Taylor series of f at a.

2. (a) Taking for granted that $\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$ for all x, find the Taylor series of $x\sin(x^3)$ at 0.

   (b) What is the radius of convergence of the power series you found in part (a)?

   (c) Let $f(x) = x\sin(x^3)$. What is $f'''(0)$? $f^{(4)}(0)$?

Theorem. If the power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ has radius of convergence R where $R > 0$ or $R = \infty$, then the function $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$ is differentiable on the interval $(a-R, a+R)$ and

(a) $f'(x) = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1}$.

(b) $\int f(x)\, dx = C + \sum_{n=0}^{\infty} \frac{c_n}{n+1}(x-a)^{n+1}$.

The power series in (1) and (2) both have radius of convergence R. (Note: Although the radius of convergence remains unchanged, the interval of convergence may change.)

3. (a) Find a power series representation of $\arctan(5x)$ centered at 0.

   (b) What is the radius of convergence of the power series you found in part (a)?

4. In each part, find a power series that has the given interval of convergence. (Hint: If you get stuck, try finding the interval of convergence of $\sum_{n=1}^{\infty} \frac{x^n}{n}$.)

   (a) $(-6, 0)$.

   (b) $(-1, 3)$.

   (c) Challenge: $[-1, 3)$.

Series Problems

1. Decide whether the following series converge or diverge. Explain your reasoning.

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{2n^5 + 500n^4 + n^3}{n^7 + 200n^6}$.

   (b) $\displaystyle\sum_{n=100}^{\infty} \frac{\sin n}{n^2}$.

2. Use a second degree Taylor polynomial to approximate $33^{1/5}$.

3. We define a function $f(x)$ by setting $f(x) = \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{\sqrt{n}2^n}$ for those x for which the series converges.

(a) Find the radius of convergence.

(b) Write a power series representation of $f'(x)$, the first derivative of f. Use it to find a series for $f'(1)$.

(c) Write out the first three non-zero terms of your series for $f'(1)$. At $x = 1$, is the function f increasing or decreasing? Explain.

Differential Equations: An Introduction to Modeling

In #1 - #8, write a differential equation that reflects the situation. Include an initial condition if the information is given.

1. The population of a certain country increases at a rate proportional to the population size. Let $P = P(t)$ be the population at time t.

2. A snowball melts at a rate proportional to its surface area. At time 0, the snowball has a radius of 10 cm. Let $r = r(t)$ be the radius of the snowball at time t.

3. A yellow rubber duck is dropped out of the window of an apartment building at a height of 80 feet. Let $s = s(t)$ be the height of the duck above the ground at time t. (Gravity is the acceleration -32 ft/s^2.)

4. Ferdinand is trying to fill a bucket from a faucet. Unfortunately, he doesn't realize that there is a small hole in the bottom of the bucket. Water flows in to the bucket from the faucet at a constant rate of .75 quarts per minute, and it flows out of the hole at a rate proportional to the amount of water $W(t)$ already in the bucket (due to the increased water pressure).

5. A drug is being administered to a patient at a constant rate of c mg/hr. The patient metabolizes and eliminates the drug at a rate proportional to the amount in his body. Let $M = M(t)$ be the amount (in mg) of medicine in the patient's body at time t, where t is measured in hours.

6. \$6000 is deposited in a bank account. The account has a nominal annual interest rate of 2%, compounded continuously. There are no deposits and no withdrawals. Let $M = M(t)$ be the amount of money in the account at time t, where t is measured in years.

7. \$6000 is deposited in a bank account. The account has a nominal annual interest rate of 2%, compounded continuously. Money is being withdrawn at a rate of \$500 per year.[1] Let $M = M(t)$ be the amount of money in the account at time t, where t is measured in years.

8. A rumor spreads at a rate proportional to product of the number of people who have heard it and the number who have not. In a town of N people, suppose 1 person originates the rumor at time $t = 0$. Let $y = y(t)$ be the number of people who have heard the rumor at time t.

   What does this model imply about the number of people who eventually have heard the rumor?

The following problems are about solutions to differential equations.

9. Which of the following is a solution to $\frac{dy}{dx} = y$?

   (a) $y = \frac{x^2}{2} + C$.

   (b) $y = e^x + C$.

   (c) $y = Ce^x$.

10. Give two solutions to $\frac{dy}{dx} = 5y$. What is the general solution?

11. Give two solutions to to $\frac{dy}{dx} = 5x$. What is the general solution?

[1] In reality, you cannot withdraw money continuously from the bank, but it's convenient to use a continuous model.

Slope Fields

1. Draw the slope fields for the following differential equations:

    (a) $\frac{dy}{dt} = 1$.

    (b) $\frac{dy}{dt} = t$.

    (c) $\frac{dy}{dt} = y$.

    (d) $\frac{dy}{dt} = \frac{-t}{y}$.

2. Draw the slope field for the differential equation $\frac{dy}{dt} = y - 1$. Sketch two solutions to the equation.

3. Which of the following is a solution to $\frac{dy}{dt} = y - 1$?

   (a) $y = Ce^t$

   (b) $y = Ce^t - t$

   (c) $y = Ce^{-t} - 1$

   (d) $y = Ce^t - 1$

   (e) $y = Ce^t + 1$

4. Which of the following is a solution to $y'' - y' - 6y = 0$?

   (a) $y = Ce^t$.

   (b) $y = C \sin 2t$.

   (c) $y = 5e^{3t} + e^{-2t}$.

   (d) $y = e^{3t} - 2$.

Separation of Variables / Mixing Problems

1. Find the general solution of the differential equation $\frac{dM}{dt} = 2.4 - .2M$. (Such a differential equation came up, for instance, when we modeled the amount of medicine in a patient's body.)

2. Last time, we solved the differential equation $\frac{dy}{dt} = -\frac{t}{y}$ by drawing the slope field, guessing the solution, and checking it. Now, solve the differential equation using separation of variables.

3. Solve the differential equation $\frac{dy}{dt} = e^{-t-y}$, and find the particular solution satisfying the initial condition $y(0) = 1$.

4. Solve the differential equation $y' = 2y - 6$.

5. Which of the following differential equations are separable? (You need not solve.)

   (a) $\frac{dy}{dt} = t + y$.

   (b) $\frac{dy}{dt} = \frac{y}{\sin t}$.

   (c) $\frac{dy}{dt} = \frac{\sin t}{y} + t$.

6. A 20-quart juice dispenser in a cafeteria is filled with a juice mixture that is 10% mango and 90% orange juice. A pineapple-mango blend (40% pineapple and 60% mango) is entering the dispenser at a rate of 4 quarts an hour and the well-stirred mixture leaves at a rate of 4 quarts an hour. Model the situation with a differential equation whose solution, $M(t)$, is the amount of mango juice in the container at time t. ($t = 0$ is the time when the pineapple-mango blend starts to enter the dispenser.)

7. Suppose that, in the previous problem, the mixture was leaving at a rate of 5 quarts per hour rather than 4 quarts per hour. Model the new situation.

Second-Order Homogeneous Differential Equations with Constant Coefficients

1. Suppose $f(t)$ and $g(t)$ are both solutions to the differential equation $y''+by'+cy=0$. Is $C_1f(t)+C_2g(t)$ a solution as well?

2. Can you guess solutions of $y''=y$? Try to guess two solutions that are not just multiples of each other.

3. Can you guess solutions of $y''=4y$? Try to guess two solutions that are not just multiples of each other.

4. Solve $y''-y'=6y$.

5. Solve $y'' + 5y' + 4y = 0$ where $y(0) = 1$ and $y'(0) = 2$.

6. Solve $y'' - 4y' + 4y = 0$.

7. Show that, if the characteristic equation $y'' + by' + cy = 0$ has one repeated root r, then $y = te^{rt}$ is a solution to $y'' + by' + cy = 0$.

Second-Order Homogeneous Differential Equations with Constant Coefficients

1. Solve $y'' - 6y' + 9y = 0$.

2. Solve $y'' + y = 0$.

3. Solve $y'' - 2y' + 5y = 0$.

4. (a) Solve $y'' + 2y' + 4y = 0$ with initial conditions $y(0) = 1$ and $y'(0) = 0$.

(b) Interpret part (a) in terms of a vibrating spring. What is happening to the spring as time goes on?

5. Which of the following differential equations has periodic solutions? What is the period?

(a) $y'' + 2y' - 3y = 0$.

(b) $y'' + 2y + 3y = 0$.

(c) $y'' + 4y' = 0$.

(d) $y'' + 4y = 0$.

(e) $y'' - 4y = 0$.

Does this agree with your interpretation of the differential equations in terms of vibrating springs?

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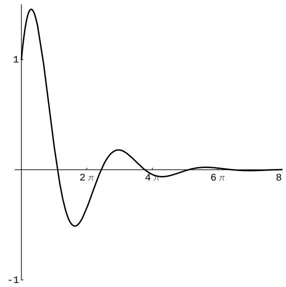
   (d) $y'' + 4y = 0$.

   (e) $y'' - 4y = 0$.

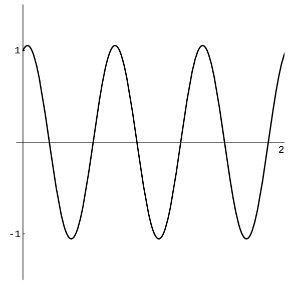
   Does this agree with your interpretation of the differential equations in terms of vibrating springs?

2. A spring with a mass of 5 kg has a natural length of 6 cm. A 20 N force is required to compress it to a length of 5 cm. If the spring is stretched to a length of 7 cm and released, find the position of the mass at time t. Sketch a graph of the position vs. time.

3. Match each differential equation with the graph of its solution. In each case, the differential equation has initial conditions $y(0) = 1$, $y'(0) = 1$.



(1)



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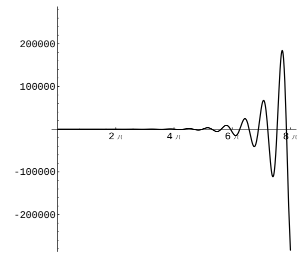

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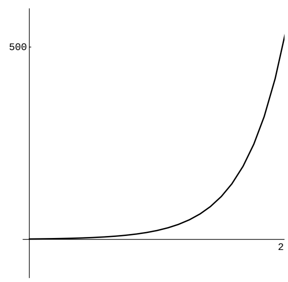

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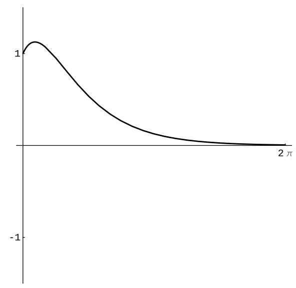


(5)

(a) $y'' + 3y' + 2y = 0.$

(b) $y'' + 9y = 0.$

(c) $y'' - 2y' + y = 0.$

(d) $y'' - y' + 10y = 0.$

(e) $y'' + \frac{1}{2}y' + \frac{5}{8}y = 0.$

4. Solve the differential equation $y'' + y = 0$ with initial conditions $y(0) = 0$ and $y'(0) = 1$.

Systems of Differential Equations

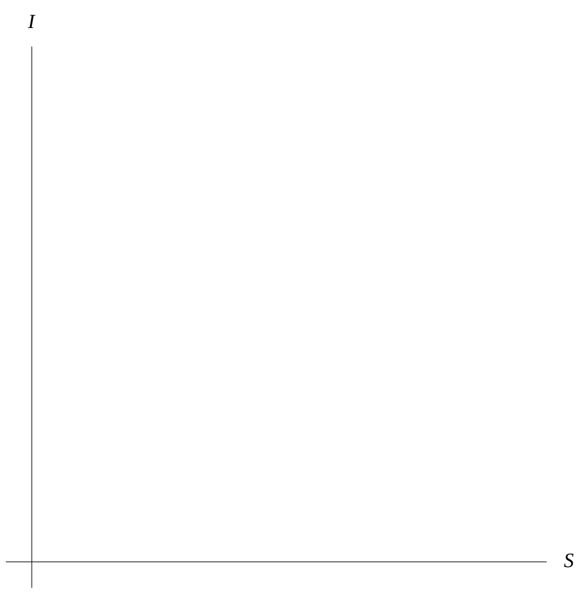
We've used systems of differential equations to model interaction between species. Systems can also be used to model disease epidemics.

Suppose that there is a large population of people, and some of the people have a fatal disease. This disease is infectious, so anybody who doesn't have the disease is susceptible to getting it. Let $I(t)$ be the number of people infected at time t, and let $S(t)$ be the number of people who are susceptible at time t.

1. How could you model this situation with a system of differential equations? You may ignore birth and death, except for death due to the disease, which you should include. (There are many many different answers; when in doubt, opt for simplicity.)

2. Using common sense, find the equilibrium points in this model. (You do not need to use the differential equations you found in #**??**; just think about the situation.)

3. Using common sense, sketch some typical phase trajectories in the phase plane.



4. A reasonable system for the situation described is:

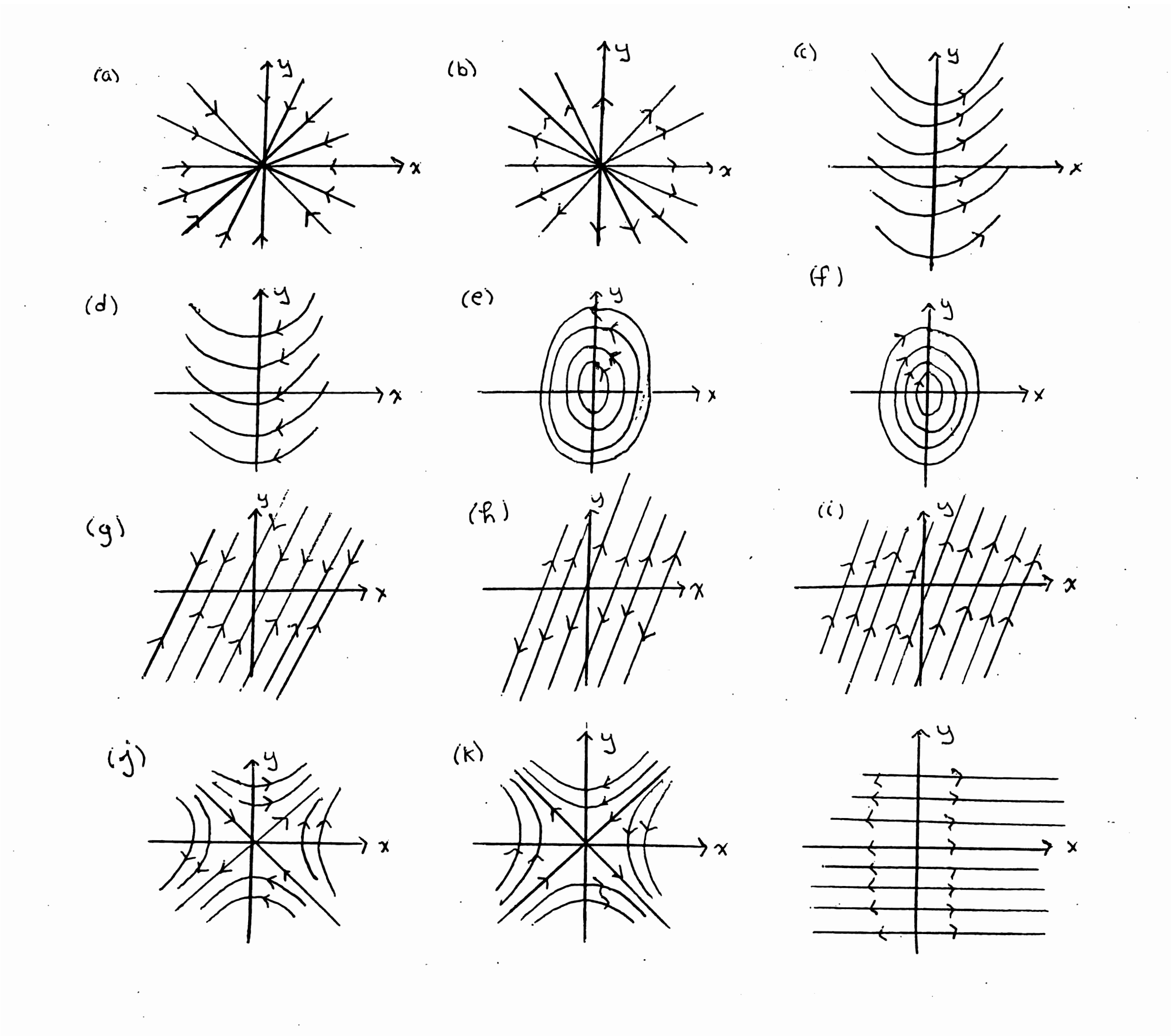
$$\begin{aligned}\frac{dS}{dt} &= -0.001IS \\ \frac{dI}{dt} &= 0.001IS - 0.1I\end{aligned}$$

Sketch the phase portrait for this system. (Be sure to draw the nullclines and equilibrium points.)

5. If the population starts with 50 infected people and 200 susceptible people, what will happen in the long run?

Systems of Differential Equations

The problems refer to these diagrams:

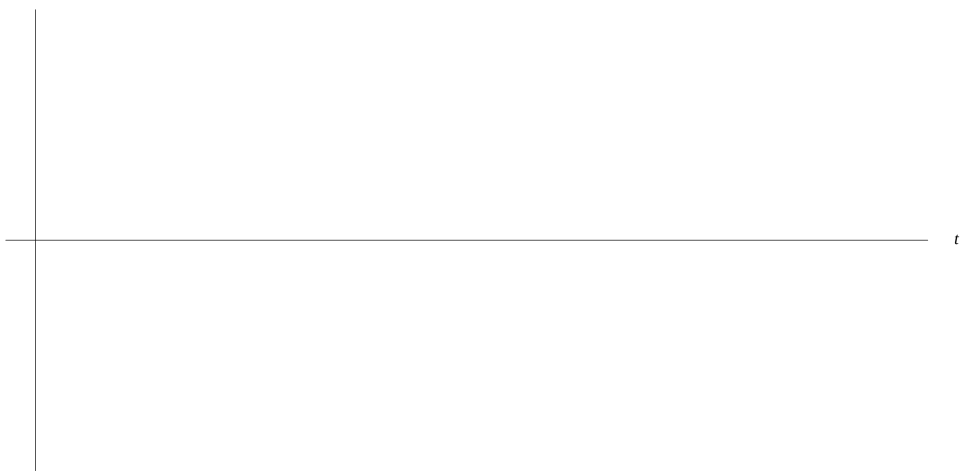


1. Find the diagram which matches the system.

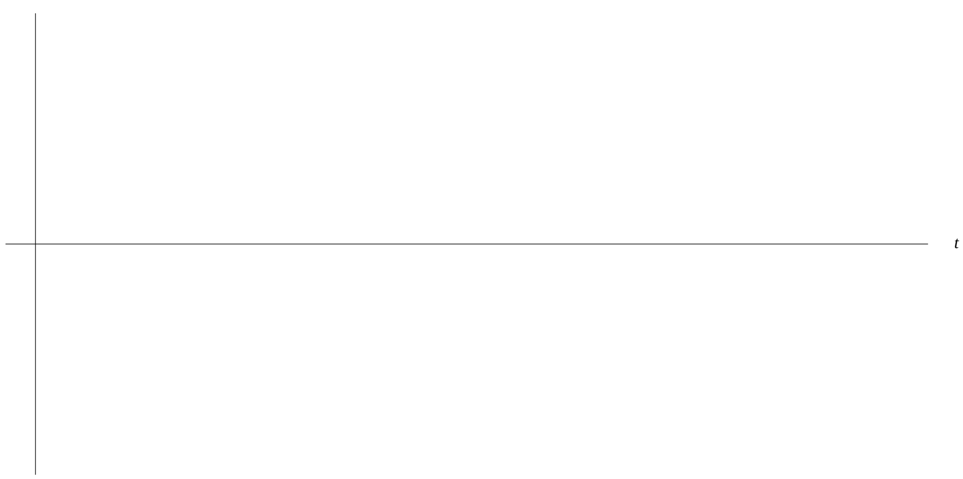
   (i) $\frac{dx}{dt} = y$ and $\frac{dy}{dt} = -2x$.

   (ii) $\frac{dx}{dt} = y$ and $\frac{dy}{dt} = 3y$.

2. (i) Suppose that the system $\frac{dx}{dt} = f(x, y)$ and $\frac{dy}{dt} = g(x, y)$ has phase portrait (e). Sketch possible graphs of $x(t)$ and $y(t)$, assuming $x(0) = 0$ and $y(0) = 3$.



   (ii) Do the same thing for (g).



3. Solve the system $\frac{dx}{dt} = y$ and $\frac{dy}{dt} = 3y$ with initial conditions $x(0) = 0$ and $y(0) = 3$.

4. Solve the system $\frac{dx}{dt} = y$ and $\frac{dy}{dt} = -2x$ with initial conditions $x(0) = 0$ and $y(0) = 3$.

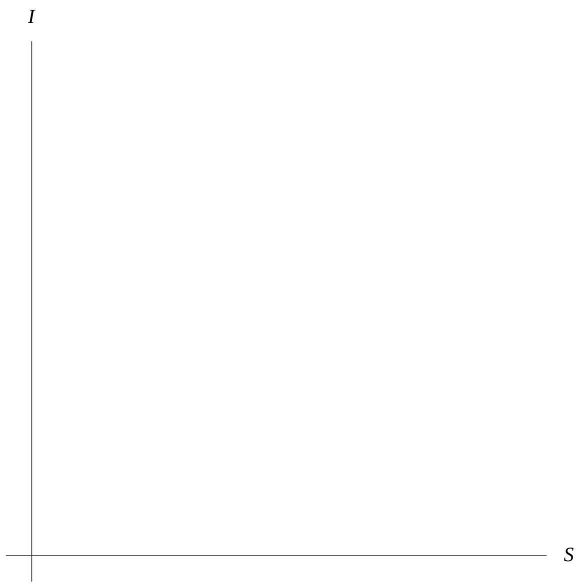
5. Find a system of differential equations whose phase portrait looks like (l), the last diagram.

Measles

We'll look at the system

$$\frac{dS}{dt} = -IS + 50$$
$$\frac{dI}{dt} = IS - 10I$$

1. Do a qualitative phase plane analysis of this system. (You should draw equilibrium points, nullclines, and the direction of the trajectories in each region.)



2. Based on your phase plane analysis, what do you think the trajectories look like? Sketch a possible trajectory on your diagram if $S(0) = 5$ and $I(0) = 20$.

3. Using your trajectory, sketch a possible graph of $I(t)$ if $S(0) = 5$ and $I(0) = 20$.

4. This graph shows the number of cases of measles in 2 week periods in London from 1944 to 1966. Does our system give the same qualitative behavior?