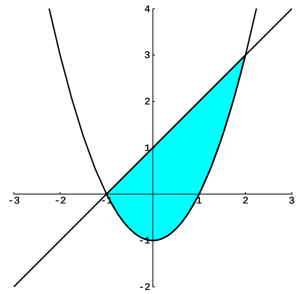
1. *The definite integral is defined to be*

   (a) a limit of Riemann sums.
   (b) the difference in the evaluation of an antiderivative at the endpoints of the interval.
   (c) a signed area.
   (d) all of the above.

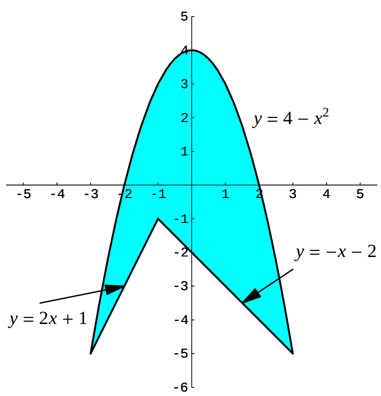
   Solution. $\boxed{\text{(a)}}$. We often *interpret* the definite integral as signed area, and we *compute* the definite integral using (b). However, neither of those is the *definition* of the definite integral.

2. *Write an integral or a sum and/or difference of integrals that gives the area enclosed by the graphs of* $y = x^2 - 1$ *and* $y = x + 1$.

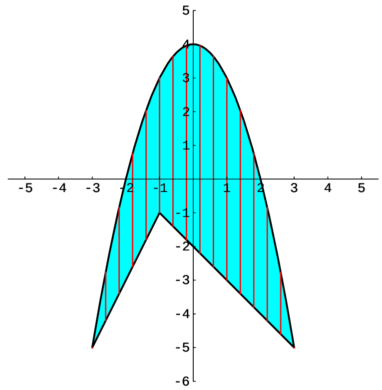


   Solution. The simplest answer is $\boxed{\int_{-1}^{2} [(x+1) - (x^2 - 1)]\ dx}$.

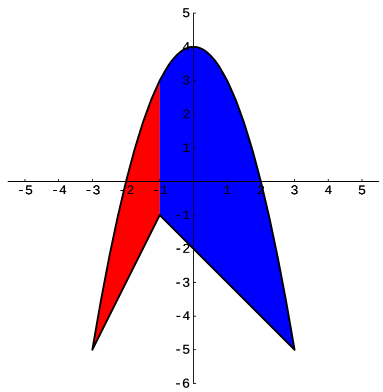
3. *Find the area of the following region.*



   Solution. According to our general theme, we should try to slice, approximate, sum, and take a limit. Let's try slicing vertically:

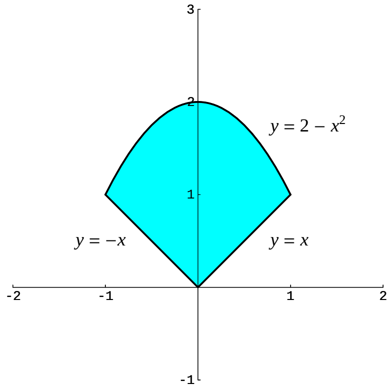


Notice that there are two "types" of slices in this picture. When x is in the interval $[-3,-1]$, the slices have height $(4-x^2)-(2x+1)$. When x is in the interval $[-1,3]$, the slices have height $(4-x^2)-(-x-2)$. We should really think about these two cases separately:



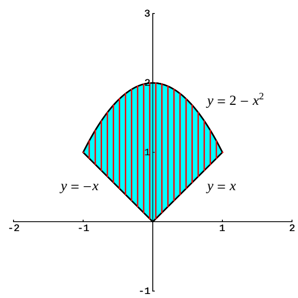
We'll find the area of the red piece and the blue piece separately and then just add those together. As we've already said, in the red piece, the slices have height $(4-x^2)-(2x+1)$, so the area of the red piece is $\int_{-3}^{-1}[(4-x^2)-(2x+1)]\ dx$. In the blue piece, the slices have height $(4-x^2)-(-x-2)$, so the area of the blue piece is $\int_{-1}^{3}[(4-x^2)-(-x-2)]\ dx$. Thus, the total area is $\boxed{\int_{-3}^{-1}[(4-x^2)-(2x+1)]\ dx+\int_{-1}^{3}[(4-x^2)-(-x-2)]\ dx}$. We can use the Fundamental Theorem of Calculus to evaluate these integrals, and we get $\boxed{\frac{38}{3}}$.

1. *Find the area of the region shown. (You may leave your answer as an integral or sum/difference of integrals.)*

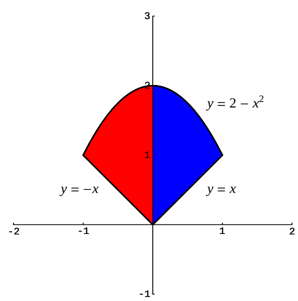


Solution. Remember that the basic idea in every integration problem is: slice, approximate, sum, and take a limit. What this boils down to is finding an expression for what's going on with the k-th slice. In this case, we're looking for area, so we want to slice the region and then approximate the area of the k-th slice.

There is a slight hitch though. Let's imagine that we slice the region vertically:

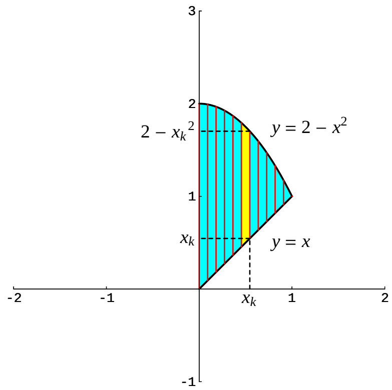


The problem is that we can't write a good description for the k-th slice. If the slice is on the left side of the y-axis, then its height would be about $(2 - x_k^2) - (-x_k)$. If the slice is on the right side, then its height is about $(2 - x_k^2) - x_k$. The solution to this problem is to break the region into two pieces and find their areas separately.



Actually, since the picture is symmetric, we can see right away that the red area is equal to the blue area. So, we can get by just finding the blue area and multiplying that by 2 to get the final answer.

So, let's focus on the right side. As usual, we slice into n pieces of equal width Δx.



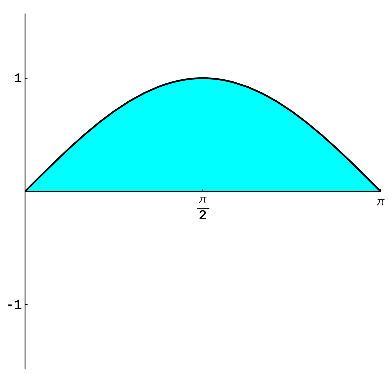
We can pretend that the k-th slice is a rectangle of width Δx and height $(2 - x_k^2) - x_k$, so we end up with

$$\boxed{\text{area of the } k\text{-th slice} \approx [(2 - x_k^2) - x_k]\Delta x}$$

Then we know that the limit of Riemann sums is $\lim_{n\to\infty} \sum_{k=1}^{n} [(2 - x_k^2) - x_k]\Delta x$, which is equal to the integral $\int_0^1 [(2 - x^2) - x]\ dx$.

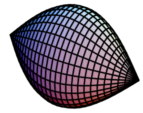
Finally, remember that this expression only represents half of the original area, so our final answer is $\boxed{2\int_0^1 [(2 - x^2) - x]\ dx}$.

2. Here is one loop of the sine curve.

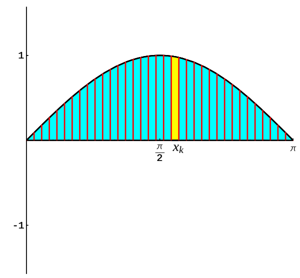


(a) *If you rotate this region about the x-axis, what shape do you get? What is its volume? (You do not need to evaluate your integral.)*

Solution. The solid looks like a football.



To find its volume, we slice and figure out what's happening with the k-th slice.



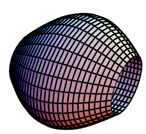
The k-th slice is very close to being a disk with radius $\sin x_k$ and thickness Δx, so

$$\boxed{\text{volume of } k\text{-th slice} \approx \pi(\sin x_k)^2 \Delta x}$$

Adding these up and taking the limit gives $\lim_{n\to\infty} \sum_{k=1}^{n} \pi(\sin x_k)^2 \Delta x$, which is equal to the integral $\boxed{\int_0^\pi \pi(\sin x)^2 \, dx}$.

(b) *If you rotate the region about the line $y = -1$, what shape do you get? What is its volume? (You do not need to evaluate your integral.)*

Solution. This looks like a bead.



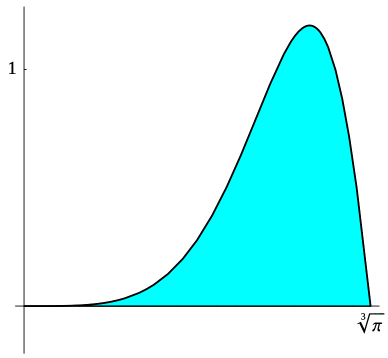
Again, to find its volume, we slice and figure out what's happening with the k-th slice. Now, the k-th slice isn't a disk any more; it's close to being a washer (disk with a round hole cut out). The outer radius of the washer is $\sin x_k + 1$, the inner radius is 1, and the thickness is Δx. Therefore,

$$\boxed{\text{volume of } k\text{-th slice} \approx \pi[(\sin x_k + 1)^2 - 1^2]\Delta x}$$

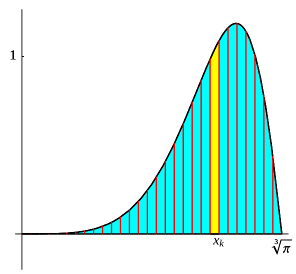
Adding these up and taking the limit gives us $\lim_{n\to\infty} \sum_{k=1}^{n} \pi[(\sin x_k + 1)^2 - 1^2]\Delta x$, which is equal to the integral $\boxed{\int_0^\pi \pi[(\sin x + 1)^2 - 1] \, dx}$.

More on Volumes

1. *This is the curve $y = x \sin x^3$. If we rotate this region about the y-axis, what is the volume of the resulting solid? (Once you get an integral, try to evaluate it.)*



Solution. We may either slice the region horizontally or vertically. Slicing horizontally will give us washers, but to find the inner radius, we would need to solve $y = x \sin x^3$ for x. We don't really know how to do that, so we had better use vertical slices instead.



When we rotate the k-th slice about the y-axis, we get a hollow tube (like a paper towel tube). The radius of this tube is x_k, the height is $x_k \sin x_k^3$, and the thickness is Δx. Therefore, the volume of the tube is approximately $2\pi x_k(x_k \sin x_k^3)\Delta x = 2\pi x_k^2 \sin x_k^3 \Delta x$.

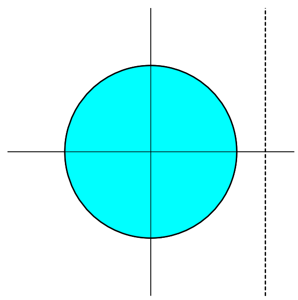
To approximate the volume of the whole solid, we have to sum up the volumes of all of the slices (slice 1 through slice n), which gives $\sum_{k=1}^{n} 2\pi x_k^2 \sin x_k^3 \Delta x$. To make the approximation accurate, we need to use more and more slices, so we take the limit as $n \to \infty$. So, our answer is $\lim_{n\to\infty} \sum_{k=1}^{n} 2\pi x_k^2 \sin x_k^3 \Delta x$, which is the integral $\int_0^{\sqrt[3]{\pi}} 2\pi x^2 \sin x^3 \, dx$. We can evaluate this integral using substitution. Let $u = x^3$.

Then, $du = 3x^2\ dx$. Since x goes from 0 to $\sqrt[3]{\pi}$, u goes from 0 to π. So,

$$\begin{aligned}
\int_0^{\sqrt[3]{\pi}} 2\pi x^2 \sin x^3\ dx &= \int_0^{\pi} 2\pi \sin u \frac{du}{3} \\
&= \frac{2\pi}{3} \int_0^{\pi} \sin u\ du \\
&= \frac{2\pi}{3} (-\cos u)\Big|_0^{\pi} \\
&= \frac{2\pi}{3} (-\cos \pi + \cos 0) \\
&= \boxed{\frac{4\pi}{3}}
\end{aligned}$$

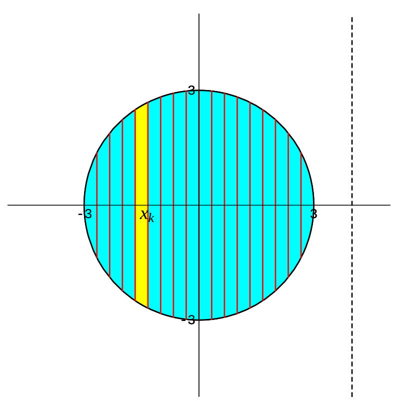
2. *How can you describe a bagel as a solid of revolution? (That is, what sort of region would you rotate, and what line would you rotate it about?)*

   Solution. We can get a bagel by rotating a disk around a line, something like this (rotate the disk around the dotted line):



3. *The disk of radius 3 centered at the origin is rotated about the line $x = 4$. Find the volume using vertical slices. (It is also possible to do it using horizontal slices, and you might want to try that for extra practice.)*

   Solution. Here are the slices.



The equation of the circle is $x^2 + y^2 = 9$, or $y = \pm\sqrt{9-x^2}$. $y = \sqrt{9-x^2}$ is the equation of the top half of the circle, and $y = -\sqrt{9-x^2}$ is the equation of the bottom half of the circle.

Rotating the k-th slice gives (approximately) a paper towel tube with thickness Δx, radius $4 - x_k$, and height $2\sqrt{9-x_k^2}$. So, the volume of the k-th piece is approximately $2\pi(4-x_k)(2\sqrt{9-x_k^2})\Delta x = 4\pi(4-x_k)\sqrt{9-x_k^2}\Delta x$.

Adding these up and taking the limit, we get the integral $\displaystyle\int_{-3}^{3} 4\pi(4-x)\sqrt{9-x^2}\ dx = 4\pi\int_{-3}^{3}(4-x)\sqrt{9-x^2}\ dx$. To evaluate this integral, let's first multiply it out a little:

$$\begin{aligned} 4\pi\int_{-3}^{3}(4-x)\sqrt{9-x^2}\ dx &= 4\pi\left[\int_{-3}^{3} 4\sqrt{9-x^2}\ dx - \int_{-3}^{3} -x\sqrt{9-x^2}\ dx\right] \\ &= 16\pi\int_{-3}^{3}\sqrt{9-x^2}\ dx + 4\pi\int_{-3}^{3} x\sqrt{9-x^2}\ dx \end{aligned}$$

Now, we have two integrals to evaluate. We don't know an antiderivative of $\sqrt{9-x^2}$, but $y = \sqrt{9-x^2}$ is just the graph of the top half of the circle in our picture. So, $\displaystyle\int_{-3}^{3}\sqrt{9-x^2}$ is the area of the top half of the circle, and we know that a circle of radius 3 has area 9π. So, $\displaystyle\int_{-3}^{3}\sqrt{9-x^2}\ dx = \frac{9\pi}{2}$.

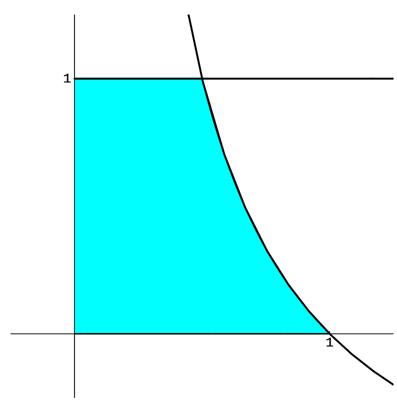
The second integral $\displaystyle\int_{-3}^{3} x\sqrt{9-x^2}\ dx$ can be done using substitution: let $u = 9 - x^2$. Then $du = -2x\ dx$. So, we can rewrite the integrand as $\int -\sqrt{u}\frac{du}{2}$. Since x goes from -3 to 3, u goes from 0 to 0, and $\displaystyle\int_0^0 -\frac{\sqrt{u}}{2}\ du = 0$.

So, our final answer is $16\pi \cdot \frac{9\pi}{2} = \boxed{72\pi^2}$.

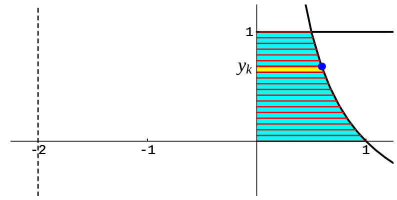
4. Let $\mathcal{R}$ be the region enclosed by the x-axis, the y-axis, $y = 1$, and $y = \frac{1}{x} - 1$.

   (a) *Find the volume generated when $\mathcal{R}$ is rotated about the line $x = -2$.*

   Solution. Here is the region $\mathcal{R}$.



As always, we should first decide whether we want to use vertical or horizontal slices. In this case, if we use vertical slices, we will have to split the region up into where $x < \frac{1}{2}$ and $x > \frac{1}{2}$ because the slices will have different descriptions. Instead, let's use horizontal slices.



Since we're using horizontal slices, we should describe things in terms of y, so let's solve $y = \frac{1}{x} - 1$ for x in terms of y.

$$\begin{aligned} y &= \frac{1}{x} - 1 \\ y + 1 &= \frac{1}{x} \\ \frac{1}{y+1} &= x \end{aligned}$$

So, the blue point is $\left(\frac{1}{y_k+1}, y_k\right)$.

After we rotate our horizontal slice, we will end up with something that is approximately a washer (or CD) with inner radius 2, outer radius $\frac{1}{y_k+1} + 2$, and thickness Δy. So, the volume of the k-th slice is approximately $\left[\pi\left(\frac{1}{y_k+1} + 2\right)^2 - \pi \cdot 2^2\right] \Delta y$. Adding these up and taking the limit gives the integral $\int_0^1 \left[\pi\left(\frac{1}{y+1} + 2\right)^2 - \pi \cdot 2^2\right] dy$. To compute this, we'll first simplify:

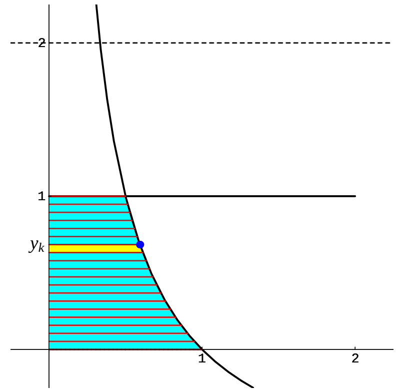
$$\begin{aligned} \int_0^1 \left[\pi\left(\frac{1}{y+1} + 2\right)^2 - \pi \cdot 2^2\right] dy &= \pi \int_0^1 \left[\left(\frac{1}{y+1} + 2\right)^2 - 2^2\right] dy \\ &= \pi \int_0^1 \left[\frac{1}{(y+1)^2} + \frac{4}{y+1} + 4 - 4\right] dy \\ &= \pi \int_0^1 \left[\frac{1}{(y+1)^2} + \frac{4}{y+1}\right] dy \end{aligned}$$

To evaluate, we'll use the substitution $u = y + 1$. Then, $du = dy$ and u goes from 1 to 2 (since y went from 0 to 1). So, we have

$$\begin{aligned} \text{volume} &= \pi \int_1^2 \left(\frac{1}{u^2} + \frac{4}{u}\right) du \\ &= \pi \left(-u^{-1} + 4\ln|u|\right)\Big|_1^2 \\ &= \pi\left[\left(-\frac{1}{2} + 4\ln 2\right) - (-1)\right] \\ &= \boxed{\pi\left(\frac{1}{2} + 4\ln 2\right)} \end{aligned}$$

(b) *Find the volume generated when $\mathcal{R}$ is rotated about the line $y = 2$.*

Solution. For the same reason as in part (a), we'll use horizontal slices.



As we figured out in part (a), the blue point has coordinates $\left(\frac{1}{y_k+1}, y_k\right)$.

After we rotate our horizontal slice, we will end up with a paper towel tube with radius $2 - y_k$, height $\frac{1}{y_k+1}$, and thickness Δy. So, the volume of this slice is approximately $2\pi(2-y_k)\frac{1}{y_k+1}\Delta y = 2\pi\frac{2-y_k}{y_k+1}$. Adding these up and taking the limit gives $\lim_{n\to\infty}\sum_{k=1}^{n} 2\pi\frac{2-y_k}{y_k+1}$, which is just the integral $\int_0^1 2\pi\frac{2-y}{y+1}\, dy$.

Now, we have to actually evaluate the integral. Let's try using $u = y + 1$. Then, $du = dy$. Also, $y = u - 1$, so $2 - y = 2 - (u - 1) = 3 - u$. Since y goes from 0 to 1, u goes from 1 to 2. So,

$$
\begin{aligned}
\int_0^1 2\pi\frac{2-y}{y+1}\, dy &= 2\pi\int_0^1 \frac{2-y}{y+1}\, dy \\
&= 2\pi\int_1^2 \frac{3-u}{u}\, du \\
&= 2\pi\int_1^2 \left(\frac{3}{u}-1\right)\, du \\
&= 2\pi\left(3\ln|u| - u\right)\Big|_1^2 \\
&= 2\pi[(3\ln 2 - 2) - (3\ln 1 - 1)] \\
&= \boxed{2\pi(3\ln 2 - 1)}
\end{aligned}
$$

Integration by Parts

Evaluate the following integrals.

1. $\int xe^x \, dx.$

2. $\int x \ln x \, dx.$

3. $\int_1^e \ln x \, dx.$

4. $\int_0^1 \arctan x \, dx.$

5. $\int x^2 \cos 2x \, dx.$

6. $\int e^x \cos x \; dx$.

7. $\int \cos \sqrt{x} \; dx$.

8. You are given the following information about an unknown function $g(x)$:

$$\int_1^2 \frac{g(u)}{u} \; du = 3, \int_1^2 g(u) \; du = 4, \int_1^4 g(u) \; du = 5, g(1) = 2, g(2) = -2.$$

(a) Evaluate $\int_1^2 (\ln x) g'(x) \; dx$.

(b) Evaluate $\int_1^2 x g(x^2) \; dx$.

9. $\int \sin 5x \sin 3x \; dx$.

Partial Fractions

1. Which of the following is easiest to integrate?

   (a) $\displaystyle\int \frac{5x-4}{x^2-x-2}\,dx.$

   (b) $\displaystyle\int \frac{5x-4}{(x-2)(x+1)}\,dx.$

   (c) $\displaystyle\int \frac{5x}{x^2-x-2}\,dx - \int \frac{4}{x^2-x-2}\,dx.$

   (d) $\displaystyle\int \frac{3}{x+1}\,dx + \int \frac{2}{x-2}\,dx$

   How do the four choices relate to each other?

2. Evaluate the following integrals.

   (a) $\displaystyle\int \frac{1}{y^2-4}\,dy.$

   (b) $\displaystyle\int \frac{5x-7}{x^2-3x+2}\,dx.$

3. Write down the form of the partial fraction expansion for the following integrals. (You don't need to actually solve for the coefficients.)

(a) $\displaystyle\int \frac{3x^2 + x + 5}{(x+1)(x+3)(x-5)}.$

(b) $\displaystyle\int \frac{x+1}{(x^2+4)(x^2+9)}.$

(c) $\displaystyle\int \frac{x^3 + 2x}{(x+4)(x+3)(x+2)^2}.$

(d) $\displaystyle\int \frac{x^2+1}{(x+1)^2(x^2+5)}.$

(e) $\displaystyle\int \frac{3x^2}{x^2+2x+1}\, dx.$

(f) $\displaystyle\int \frac{x^3+4x^2+7x}{x^2+4x+3}\, dx.$

4. Evaluate the following integrals.

(a) $\displaystyle\int \frac{x^2-x+4}{x^3+4x}.$

(b) $\displaystyle\int \frac{1}{x^3+x}\, dx.$

(c) $\displaystyle\int \frac{\sin\theta\ d\theta}{\cos^2\theta+\cos\theta-2}.$

Integration Techniques

In each problem, decide which method of integration you would use. If you would use substitution, what would u be? If you would use integration by parts, what would u and dv be? If you would use partial fractions, what would the partial fraction expansion look like? (Don't solve for the coefficients.)

1. $\displaystyle\int \frac{\cos x \; dx}{\sqrt{1+\sin x}}$.

   Solution. Use the substitution $u = 1+\sin x$ since $du = \cos x \; dx$ also appears in the integrand.

2. $\displaystyle\int (\ln x)^2 \; dx$.

   Solution. Partial fractions is definitely not right, since this is not a rational function. Substitution doesn't look so promising, so we're left with integration by parts. Since we don't know how to integrate anything involving ln, use $u = (\ln x)^2$ and $dv = dx$.

3. $\displaystyle\int e^x \sin x \; dx$.

   Solution. This is a classic integration by parts integral, where you do integration by parts twice to get back the original integral and then solve for it. You can use $u = e^x$ and $dv = \sin x \; dx$ or $u = \sin x$ and $dv = e^x \; dx$; they work equally well.

4. $\displaystyle\int \frac{x}{x^2-1} \; dx$.

   Solution. Since the integrand is a rational function, you could use partial fractions. But it's easier to just use substitution with $u = x^2 - 1$.

5. $\displaystyle\int xe^{x^2} \; dx$.

   Solution. Substitution with $u = x^2$ since $du = 2x \; dx$ also appears.

6. $\displaystyle\int \frac{x^2}{x^2+4x+3} \; dx$.

   Solution. The integrand is a rational function, and we can factor the denominator pretty easily, so partial fractions is a good choice. Since the integrand is an improper fraction, we start by rewriting it: $\frac{x^2}{x^2+4x+3} = \frac{x^2+4x+3-4x-3}{x^2+4x+3} = 1 - \frac{4x+3}{x^2+4x+3}$. The denominator factors as $(x+1)(x+3)$, so the partial fraction expansion has the form $1 + \frac{A}{x+1} + \frac{B}{x+3}$.

7. $\displaystyle\int \frac{e^t}{1+e^t} \; dt$.

   Solution. Use substitution with $u = 1+e^t$ since $du = e^t \; dt$ also appears.

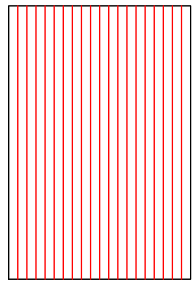
8. $\displaystyle\int \arcsin x \; dx$.

Solution. Partial fractions is definitely out, and there's not much to substitute, so use integration by parts with $u = \arcsin x$ and $dv = dx$.

1. A seaside village, Playa del Carmen, is in the shape of a rectangle 4 miles wide and 6 miles long. The sea lies along a 6-mile long side. People prefer to live near the water, so the density of people is given by $\rho(x) = 10000 - 800x$ people per square mile, where x is the distance from the seaside. We would like to find the population of the village.

   (a) *Show in a sketch how to slice up the region.*

   Solution. We'll slice the region parallel to the sea, like this:

   The reason we do it this way is in part (c).

   (b) *What is the area of the k-th slice?*

   Solution. We'll call the width of each slice Δx. Then, the area of the k-th slice is $6\Delta x$.

   (c) *What is the approximate population in the k-th slice?*

   Solution. This is really the key to the problem. Because of the way we sliced, the population density within each slice is close to being constant. Therefore, we can approximate the population as (population density) times (area), or $\rho(x_k) \cdot 6\Delta x = 6\rho(x_k)\Delta x$.

   (d) *Write a general Riemann sum to estimate the total population of the city.*

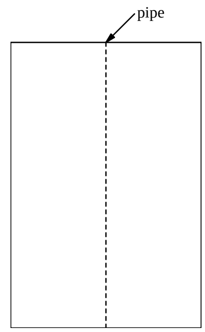
   Solution. To approximate the total population, we just add up the approximate population in all slices, which gives $\sum_{k=1}^{n} 6\rho(x_k)\Delta x$.

   (e) *Find a definite integral expressing the population of the village.*

   Solution. The assumption we made in our approximation was that the population density in each slice was constant. This assumption becomes more accurate as we use more slices, since each slice will be thinner. Therefore, to get the actual population, we should take the limit as $n \to \infty$. We know that $\lim_{n\to\infty} \sum_{k=1}^{n} 6\rho(x_k)\Delta x$ is the same as the integral $\boxed{\int_0^4 6\rho(x)\ dx.}$

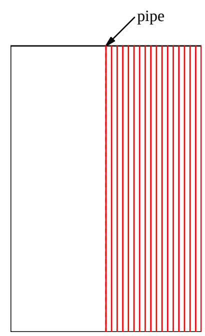
2. *A rectangular plot of farm land is* 300 *meters by* 200 *meters. A straight irrigation pipe* 300 *meters long runs down the center of the plot, dividing it in half lengthwise. The farmer's yield decreases as the distance from the irrigation pipe increases. Suppose that the yield is given by $\rho(x)$ grams per square meter, where x is the distance in meters from the irrigation pipe. Write an integral giving the total yield from the plot.*

   Solution. Here's a sketch.



First, because of symmetry, the yield in the left half of the plot is the same as the yield in the right half. Therefore, we only need to figure out the yield in one half; let's do the right half.

Since the yield depends on the distance from the irrigation pipe, we want to slices in which the distance to the irrigation pipe is approximately constant. Therefore, we'll slice parallel to the pipe:

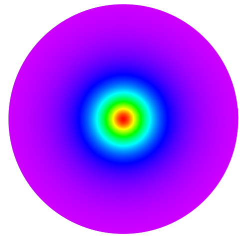


Since x represents the distance to the pipe, the leftmost slice starts where $x = 0$, and the rightmost slice ends where $x = 100$.

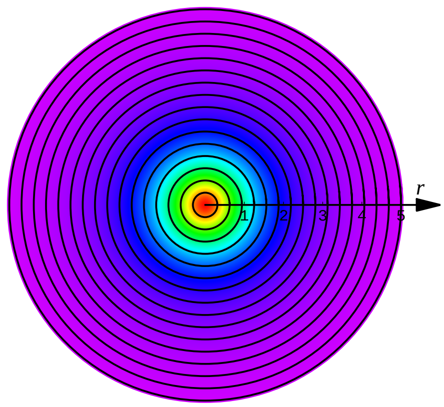
The area of each slice is $300\Delta x$. In the k-th slice, the yield rate is approximately $\rho(x_k)$ grams per square meter, so the yield is approximately $\rho(x_k) \cdot 300\Delta x = 300\rho(x_k)\Delta x$. Summing over all n slices and taking the limit as $n \to \infty$ gives us $\int_0^{100} 300\rho(x)\,dx$ as the yield for half of the plot. Therefore, the total yield is $\boxed{2\int_0^{100} 300\rho(x)\,dx}$.

3. *People in the Boston area like to live near the city center, so the population density around Boston is* $\rho(r) = \frac{36{,}000}{r^2+2r+1}$ *people per square mile, where* r *is the distance in miles to the center of Boston. Find the number of people who live within 5 miles of the center of Boston.*

Solution. The region we are talking about (within 5 miles of the center of Boston) is a disk, with the highest population density at the center. Here is a plot showing the population density in the city (red represents the highest density, and purple represents the lowest):



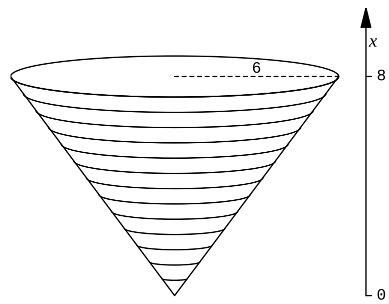
Just like in the other problems, we want to slice the region so that the population density in each slice is almost constant. In this case, we can accomplish that by slicing into concentric rings.



The population density in the k-th slice is approximately equal to $\rho(r_k)$. The area of slice k is approximately $2\pi r_k \Delta r$. So, the number of people living in the k-th slice is approximately $\rho(r_k) \cdot 2\pi r_k \Delta r = 2\pi r_k \rho(r)\Delta$. Summing over all n slices and taking the limit as $n \to \infty$ gives the integral $\boxed{\int_0^5 2\pi r \rho(r)\ dr}$.

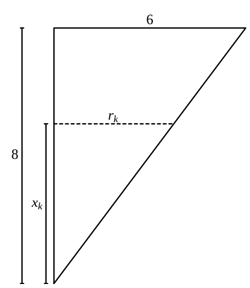
1. *A cone with height* 8 *inches and radius* 6 *inches is filled with flavored slush. When the cup is held upright with the pointed end resting on a table, the density of flavoring syrup in the cup varies with height above the table. Suppose* $\rho(x)$ *gives the number of ounces of syrup per cubic inch, where* x *is the distance from the table top. Write an integral giving the total amount of syrup in the cup.*

Solution. Since the density varies with height, we will slice the cone like this:



The k-th slice looks approximately like a disk of thickness Δx. To approximate the volume of the slice, we need to find its radius. Moreover, we should do this in terms of x_k since we're going to want to end up with an integral in terms of x.

To find the radius r_k of the k-th slice in terms of x_k, we use similar triangles:



This shows us that $\frac{6}{8} = \frac{r_k}{x_k}$, so $r_k = \frac{6}{8}x_k = \frac{3}{4}x_k$. Therefore, the volume of the k-th slice is approximately $\pi\left(\frac{3}{4}x_k\right)^2 \Delta x = \frac{9\pi}{16}x_k^2\Delta x$ cubic inches. The amount of syrup in the k-th slice is approximately $[\rho(x_k)\text{ounces per cubic inch}] \cdot \left[\frac{9\pi}{16}x_k^2\Delta x\text{cubic inches}\right] = \frac{9\pi}{16}x_k^2\rho(x_k)\Delta x$ ounces. Summing the amount in each slice gives the Riemann sum approximation $\displaystyle\sum_{k=1}^{n}\frac{9\pi}{16}x_k^2\rho(x_k)\Delta x$. Taking the limit as $n \to \infty$ gives the integral $\boxed{\displaystyle\int_0^8 \frac{9\pi}{16}x^2\rho(x)\,dx}$.

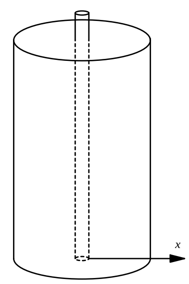
2. *Suppose the density of a planet is given by the function* $\rho(r) = \dfrac{40000}{1 + 0.0001r^3}$ *kilograms per cubic kilometer, where* r *is the distance in kilometers from the center of the planet. Find the total mass of the planet if its radius is* 8000 *km. (You do not need to evaluate your integral.)*

Solution. Since the density varies with the distance to the center, we should slice into concentric spherical shells. Each shell will have a small thickness Δr, and a good approximation for the volume of such a spherical shell is its surface area multiplied by the thickness Δr.

The k-th slice has outer radius r_k and inner radius r_{k-1}, so its volume is approximately $4\pi r_k^2 \Delta r$ km^3. The mass of this slice is approximately $[\rho(r_k)\text{kg} / \text{km}^3] \cdot [4\pi r_k^2 \Delta r \text{ km}^3] = 4\pi r_k^2 \rho(r_k)\Delta r$ kg. Adding these up gives the Riemann sum approximation $\sum_{k=1}^{n} 4\pi r_k^2 \rho(r_k)\Delta r$. Taking the limit gives the integral $\boxed{\int_0^{8000} 4\pi r^2 \rho(r)\, dr}$. If we wanted a numeric answer, we could integrate using the substitution $u = 1 + 0.0001r^3$.

3. *A cylindrical candle of height* 50 *mm and radius* 12 *mm is formed by repeatedly dipping a wick of radius* 1 *mm into hot wax and then allowing the new layer of wax to dry. The density of each new layer is slightly different, so the density of the candle varies with the distance to the wick. If $\rho(x)$ gives the density in grams per cubic mm of the wax, where x measures the distance to the wick, write an integral giving the mass of the candle.*

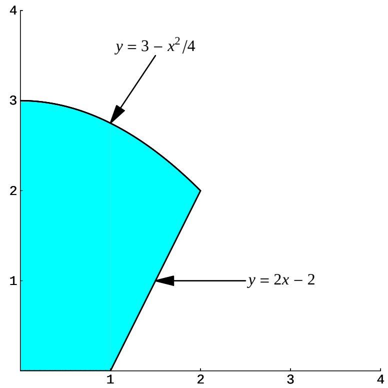
   Solution. Here is a (crude) sketch of our candle.



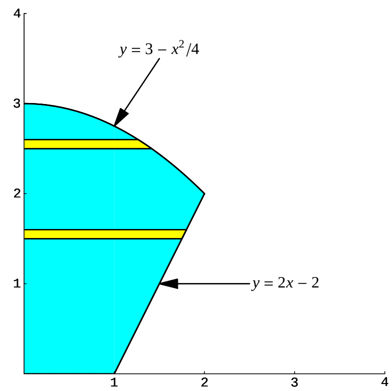
   We should slice using cylindrical shells (also known as paper towel tubes, just like what we used in some of the volumes of revolution problems).

   Each cylindrical shell has height 50 mm and thickness Δx. The k-th shell is distance x_k from the wick, so its radius is $x_k + 1$ (because we must take into account the radius of the wick). So, the volume of the k-th shell is approximately $2\pi(x_k + 1) \cdot 50 \cdot \Delta x = 100\pi(x_k + 1)\Delta x$ cubic inches. Then, its mass is approximately $[\rho(x_k)$ grams per cubic mm$] \cdot [100\pi(x_k + 1)\Delta x$ cubic mm$] = 100\pi(x_k + 1)\rho(x_k)\Delta x$ grams. Adding these up gives the Riemann sum approximation $\sum_{k=1}^{n} 100\pi(x_k + 1)\rho(x_k)\Delta x$. Taking the limit as $n \to \infty$ gives the definite integral $\boxed{\int_0^{11} 100\pi(x + 1)\rho(x)\, dx}$.

4. *We can model a muffin as a solid of revolution, obtained by rotating the following region about the y-axis. Due to a poor recipe, the chocolate chips in our muffin tend to sink to the bottom. The amount of chocolate in the muffin is given by $\rho(y) = 5 - y$ grams per cubic inch, where y represents the distance to the bottom of the muffin. Find the total amount of chocolate in the muffin.*

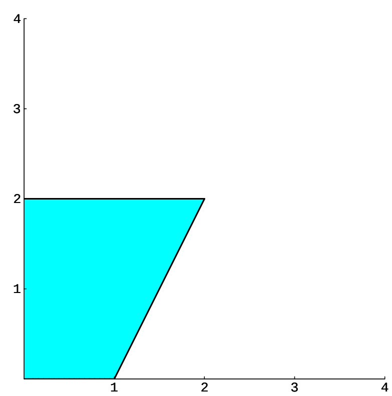


Solution. Since the chocolate density varies with the distance to the bottom of the muffin, we must slice parallel to the bottom of the muffin. Each slice is approximately a disk. Here are two representative slices:



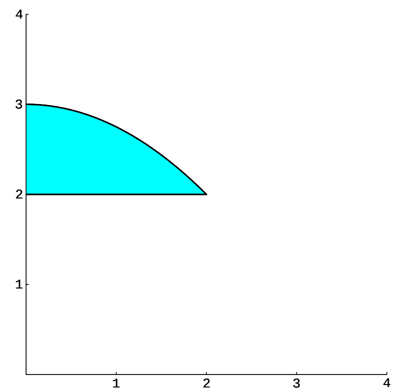
Notice that these two slices have different descriptions: the top one should be described using the curve $y = 3 - \frac{x^2}{4}$, while the bottom one should be described using the curve $y = 2x - 2$. So, we should really consider the top and bottom parts of the muffin separately.

Let's first focus on the bottom part of the muffin, which we get by rotating this region:



Here, the slices are disks, with the radius being the horizontal distance between the y-axis and $y = 2x - 2$. To find this horizontal distance, we need to solve $y = 2x - 2$ for x, which gives $x = \frac{y+2}{2}$. Therefore, the radius of the k-th slice is $\frac{y_k+2}{2}$. This means that its volume is approximately $\pi\left(\frac{y_k+2}{2}\right)^2\Delta y$, so the amount of chocolate in this slice is approximately $\rho(y_k)\cdot\pi\left(\frac{y_k+2}{2}\right)^2\Delta y = \pi\left(\frac{y_k+2}{2}\right)^2\rho(y_k)\Delta y$. Adding these up and taking the limit gives us the integral $\int_0^2 \pi\left(\frac{y+2}{2}\right)^2\rho(y)\,dy$ for the amount of chocolate in the bottom part of the muffin.

To deal with the top part of the muffin, we use exactly the same reasoning. We get the top part of the muffin by rotating this region:



We need to solve $y = 3 - \frac{x^2}{4}$ for x:

$$\begin{aligned} y &= 3 - \frac{x^2}{4} \\ \frac{x^2}{4} &= 3 - y \\ x^2 &= 4(3-y) \\ x &= 2\sqrt{3-y} \end{aligned}$$

So, the k-th slice is approximately a disk of radius $2\sqrt{3-y_k}$ and thickness Δy, which means its volume is approximately $\pi(2\sqrt{3-y_k})^2\Delta y = 4\pi(3-y_k)\Delta y$. Therefore, the amount of chocolate in this slice is approximately $4\pi(3-y_k)\rho(y_k)\Delta y$. Adding these up and taking the limit gives us the integral $\int_2^3 4\pi(3-y)\rho(y)\,dy$ for the amount of chocolate in the top part of the muffin.

So, the total amount of chocolate in the muffin is $\boxed{\int_0^2 \pi\left(\frac{y+2}{2}\right)^2\rho(y)\,dy + \int_2^3 4\pi(3-y)\rho(y)\,dy.}$

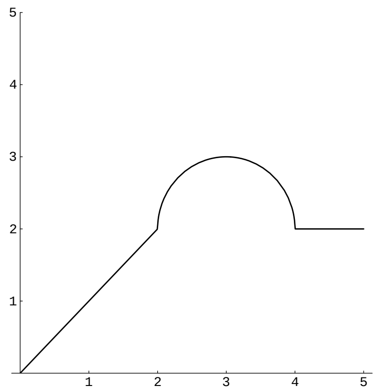
5. *You ride your bike with velocity* $v(t) = 3t^2+4t-5$ *in the time interval* $[0,3]$. *What is your average velocity?*

Solution. It is $\boxed{\frac{1}{3-0}\int_0^3 v(t)\,dt} = \frac{1}{3}\int_0^3 (3t^2+4t-5)\,dt = \frac{1}{3}(t^3+2t^2-5t)\Big|_0^3 = \boxed{10}$.

6. *The temperature outside is given by the function* $f(t)$, *where* t *represents the time since 10:00 am. How would you find the average temperature between noon and 5:00 pm?*

Solution. We want the average temperature for the interval $[2, 7]$, and that's $\boxed{\frac{1}{7-2}\int_2^7 f(t)\ dt}$.

7. *The graph of a function f is shown. The graph is made up of lines and semicircles. Find the average value of f on the interval* $[1, 5]$.



Solution. The average value of f is $\frac{1}{4}\int_1^5 f(t)\ dt$. We know that $\int_1^5 f(t)\ dt$ is the signed area of f from $t = 1$ to $t = 5$, and from the graph, we can see that this signed area is $\frac{9+\pi}{2}$. Therefore, the average value of the function is $\boxed{\frac{9+\pi}{8}}$.

1. *Write an integral that gives the length of one arch of the sine curve (so from $x = 0$ to $x = \pi$).*

   Solution. Our formula tells us that it is $\int_0^{\pi} \sqrt{1+\cos^2 x}\, dx$.

2. (a) *Does* $\int_1^{\infty} \frac{1}{x^2}\, dx$ *converge or diverge? If it converges, evaluate it.*

   Solution. We know that $\int_1^{\infty} \frac{1}{x^2}\, dx$ really means $\lim_{b\to\infty} \int_1^{b} \frac{1}{x^2}\, dx$. We can evaluate $\int_1^{b} \frac{1}{x^2}\, dx$ pretty easily: it is $-\frac{1}{x}\Big|_1^b = -\frac{1}{b} + 1 = 1 - \frac{1}{b}$. So, $\int_1^{\infty} \frac{1}{x^2}\, dx = \lim_{b\to\infty} 1 - \frac{1}{b} = \boxed{1}$.

   (b) *Does* $\int_1^{\infty} \frac{1}{x}\, dx$ *converge or diverge? If it converges, evaluate it.*

   Solution. We know that $\int_1^{\infty} \frac{1}{x}\, dx$ really means $\lim_{b\to\infty} \int_1^{b} \frac{1}{x}\, dx$, so

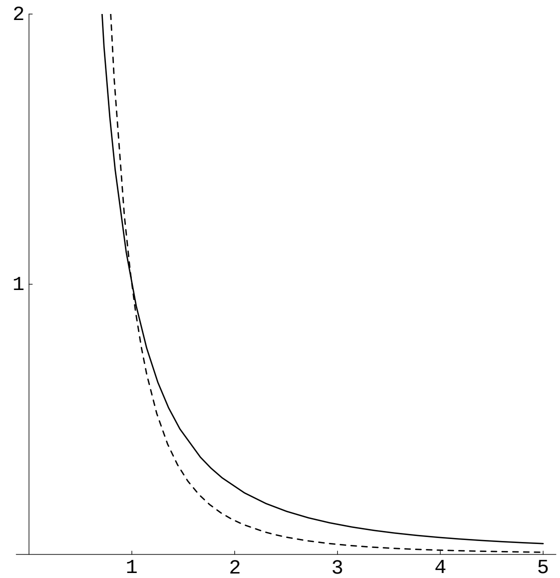
$$\begin{aligned}\int_1^{\infty} \frac{1}{x}\, dx &= \lim_{b\to\infty} \left(\ln|x| \Big|_1^b \right) \\ &= \lim_{b\to\infty} (\ln|b| - \ln 1) \\ &= \lim_{b\to\infty} \ln|b| \text{ since } \ln 1 = 0\end{aligned}$$

   But we know that $\lim_{b\to\infty} \ln|b| = \infty$, which is a form of diverging, so the improper integral $\int_1^{\infty} \frac{1}{x}\, dx$ diverges.

3. Using #2, can you conclude anything about whether the following integrals converge or diverge? (Try to figure this out without evaluating the integrals!)

   (a) $\int_1^{\infty} \frac{1}{x^3}\, dx$*?*

   Solution. Let's graph $\frac{1}{x^2}$ (the solid curve) and $\frac{1}{x^3}$ (the dashed curve):

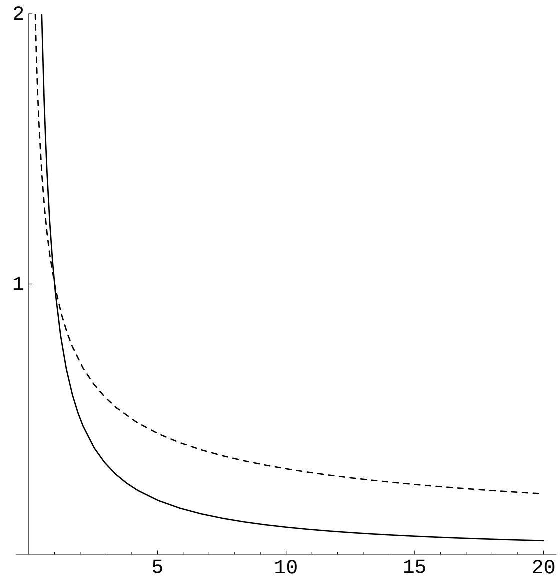


By 2(a), we know that $\int_1^\infty \frac{1}{x^2}\, dx = 1$. Graphically, we interpret this as the area under the curve $y = \frac{1}{x^2}$ to the right of $x = 1$. It is apparent from our picture that the area under $y = \frac{1}{x^3}$ to the right of $x = 1$ should be less than that, so we expect the integral $\int_1^\infty \frac{1}{x^3}\, dx$ to converge. In fact, it does:

$$\begin{aligned}
\int_1^\infty \frac{1}{x^3}\, dx &= \lim_{b\to\infty} \int_1^b \frac{1}{x^3}\, dx \\
&= \lim_{b\to\infty} \left(-\frac{1}{2}x^{-2}\Big|_1^b \right) \\
&= \lim_{b\to\infty} \left(-\frac{1}{2b^2} + \frac{1}{2} \right) \\
&= \frac{1}{2}
\end{aligned}$$

(b) $\int_1^\infty \frac{1}{x^{1/2}}\, dx$*?*

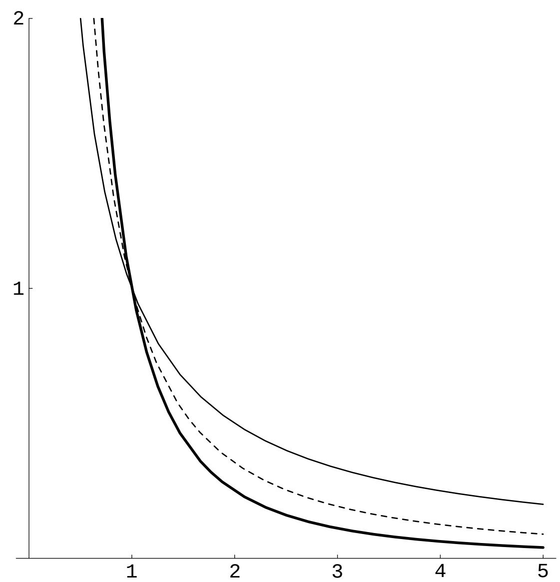
Solution. Let's graph $\frac{1}{x}$ (the solid curve) and $\frac{1}{x^{1/2}}$ (the dashed curve):



Since the graph of $\frac{1}{x^{1/2}}$ is higher than the graph of $\frac{1}{x}$ when $x \geq 1$, the area under $\frac{1}{x^{1/2}}$ to the right of $x = 1$ should be at least as big as the area under $\frac{1}{x}$ to the right of $x = 1$. The area under $\frac{1}{x}$ to the right of $x = 1$ was already infinite, so we expect $\int_1^\infty \frac{1}{x^{1/2}}\, dx$ to diverge.

(c) $\int_1^\infty \frac{1}{x^{3/2}}\, dx$?

Solution. The graph of $\frac{1}{x^{3/2}}$ lies between the graphs of $\frac{1}{x}$ and $\frac{1}{x^2}$:



Here, the thin solid graph is $y = \frac{1}{x}$, the thick solid graph is $y = \frac{1}{x^2}$, and the dashed graph is $y = \frac{1}{x^{3/2}}$. Thinking in terms of areas, we can guess that $\int_1^\infty \frac{1}{x^2}\, dx \leq \int_1^\infty \frac{1}{x^{3/2}}\, dx \leq \int_1^\infty \frac{1}{x}\, dx$.

Using our result from #2, this tells us that $1 \leq \int_1^\infty \frac{1}{x^{3/2}}\ dx \leq \infty$. Unfortunately, that doesn't give us enough information to determine whether $\int_1^\infty \frac{1}{x^{3/2}}\ dx$ converges.

In this case, comparing to $\frac{1}{x}$ and $\frac{1}{x^2}$ doesn't help, so we'll evaluate:

$$\begin{aligned}
\int_1^\infty \frac{1}{x^{3/2}}\ dx &= \lim_{b\to\infty} \int_1^b \frac{1}{x^{3/2}}\ dx \\
&= \lim_{b\to\infty} \left(-2x^{-1/2} \Big|_1^b \right) \\
&= \lim_{b\to\infty} -2b^{-1/2} + 2 \\
&= \lim_{b\to\infty} 2 - \frac{2}{\sqrt{b}} \\
&= 2
\end{aligned}$$

So, the improper integral converges to 2.

Improper Integrals

Determine whether the following integrals converge or diverge. Explain your reasoning.

1. $\displaystyle\int_{-2}^{2} \frac{x}{x^2-1}\,dx.$

   Solution. The integrand is discontinuous at $x = \pm 1$, so we know we need to split the integral. The improprieties are at -1 and 1, and each of our pieces should have at most one impropriety. So, let's split like this:

$$\int_{-2}^{2} \frac{x}{x^2-1}\,dx = \int_{-2}^{-1} \frac{x}{x^2-1}\,dx + \int_{-1}^{0} \frac{x}{x^2-1}\,dx + \int_{0}^{1} \frac{x}{x^2-1}\,dx + \int_{1}^{2} \frac{x}{x^2-1}\,dx.$$

   (You could choose a number other than 0 between -1 and 1.)

   Now, we have to evaluate each of the integrals on the right (and they are all improper). Let's first find an antiderivative of $\frac{x}{x^2-1}$ by substituting $u = x^2 - 1$:

$$\int \frac{x}{x^2-1}\,dx = \frac{1}{2}\int \frac{1}{u}\,du = \frac{1}{2}\ln|u| = \frac{1}{2}\ln|x^2-1|.$$

   Now, we start evaluating our four improper integrals using limits.

$$\begin{aligned}\int_{-2}^{-1} \frac{x}{x^2-1}\,dx &= \lim_{b\to -1^-} \int_{-2}^{b} \frac{x}{x^2-1}\,dx \\ &= \lim_{b\to -1^-} \frac{1}{2}\ln|x^2-1|\Big|_{-2}^{b} \\ &= \lim_{b\to -1^-} \left(\frac{1}{2}\ln|b^2-1| - \frac{1}{2}\ln 3\right)\end{aligned}$$

   As $b \to -1^-$, $b^2 - 1 \to 0$, so $\ln|b^2-1| \to -\infty$. Thus, this integral diverges.

   Since one of our four pieces diverges, we don't need to bother calculating the other pieces; we already know that the whole integral $\boxed{\text{diverges}}$.

2. $\displaystyle\int_{1}^{\infty} \frac{1}{x^4+2}\,dx.$

   Solution. The integrand here is very similar to $\frac{1}{x^4}$, and we know $\displaystyle\int_1^\infty \frac{1}{x^4}\,dx$ converges. This suggests that we use the Comparison Theorem.

   Notice that $0 \le \frac{1}{x^4+2} \le \frac{1}{x^4}$ for all x. Since $\displaystyle\int_1^\infty \frac{1}{x^4}\,dx$ converges, the Comparison Theorem tells us that $\displaystyle\int_1^\infty \frac{1}{x^4+2}\,dx$ also $\boxed{\text{converges}}$.

3. $\displaystyle\int_{0}^{\infty} \frac{1}{e^x+x}\,dx.$

Solution. Since we don't know how to find an antiderivative of $\frac{1}{e^x+x}$, we should use the Comparison Theorem. Since $0 \leq \frac{1}{e^x+x} \leq \frac{1}{e^x}$ for all x and you saw on your homework that $\displaystyle\int_0^\infty \frac{1}{e^x}\,dx$ converges, the Comparison Theorem tells us that $\displaystyle\int_0^\infty \frac{1}{e^x+x}\,dx$ also $\boxed{\text{converges}}$.

4. $\displaystyle\int_{-\infty}^{\infty} \sin x\ dx.$

Solution. We need to split up this integral because it has two improprieties: the $-\infty$ and the ∞. It doesn't really matter where we split it, so let's split it at 0: $\displaystyle\int_{-\infty}^{\infty} \sin x\ dx = \int_{-\infty}^{0} \sin x\ dx + \int_0^\infty \sin x\ dx.$ Let's do $\displaystyle\int_0^\infty \sin x\ dx$ first. By definition, this is $\displaystyle\lim_{b\to\infty}\int_0^b \sin x\ dx = \lim_{b\to\infty} -\cos x|_0^b = \lim_{b\to\infty}(-\cos b+1)$, which does not exist. Since this piece diverges, we know that the whole integral $\boxed{\text{diverges}}$.

5. $\displaystyle\int_1^\infty \frac{1+e^{-x}}{x}\ dx.$

Solution. We can use the Comparison Theorem: $\frac{1+e^{-x}}{x} \geq \frac{1}{x} \geq 0$ when $x \geq 1$. Since we know that $\displaystyle\int_1^\infty \frac{1}{x}\ dx$ diverges, $\displaystyle\int_1^\infty \frac{1+e^{-x}}{x}\ dx$ must $\boxed{\text{diverge}}$ as well.

6. $\displaystyle\int_1^\infty \frac{\cos^2 x}{x^2}\ dx.$

Solution. We can use the Comparison Theorem: $\frac{1}{x^2} \geq \frac{\cos^2 x}{x^2} \geq 0$. Since we know that $\displaystyle\int_1^\infty \frac{1}{x^2}\ dx$ converges, $\displaystyle\int_1^\infty \frac{\cos^2 x}{x^2}\ dx$ $\boxed{\text{converges}}$ as well.

Probability

Waiting times, shelf-lifes, and equipment failure times are often modeled by exponentially decreasing probability density functions.

1. Suppose $f(t) = 0$ for $t < 0$ and $f(t) = 0.5e^{-ct}$ for $t \geq 0$ is the probability density function for the lifetime of a particular toy (t in years).

   (a) *For what value of c is this a probability density function?*

   Solution. In order for f to be a probability density function, it must satisfy $\int_{-\infty}^{\infty} f(t)\,dt = 1$, or $\int_{-\infty}^{0} f(t)\,dt + \int_{0}^{\infty} f(t)\,dt = 1$. The first integral is 0 since $f(t) = 0$ for $t < 0$.

   If $c = 0$, then $f(t) = 0.5$ for $t \geq 0$, and the improper integral $\int_0^{\infty} f(t)\,dt$ certainly won't converge. If $c < 0$, then $f(t)$ is a positive increasing function, and again $\int_0^{\infty} f(t)\,dt$ won't converge. So, we must need to have $c > 0$.

   In this case,

$$\begin{aligned} 0.5\int_0^{\infty} e^{-ct}\,dt &= 0.5\lim_{b\to\infty}\int_0^{b} e^{-ct}\,dt \\ &= 0.5\lim_{b\to\infty} -\frac{1}{c}e^{-ct}\Big|_0^b \\ &= 0.5\lim_{b\to\infty}\left(-\frac{1}{c}e^{-cb} + \frac{1}{c}\right) \\ &= \frac{0.5}{c} \end{aligned}$$

   (Notice that, in the last step, we needed to use the fact that $c > 0$.) We want this to equal 1, so c should equal $\boxed{0.5}$.

   (b) *What is the probability that the toy lasts over one year? (Is there any way to compute this without computing an improper integral?)*

   Solution. The probability that the toy lasts over one year is given by the integral $\int_1^{\infty} f(t)\,dt$. If we want to avoid using an improper integral, we could instead calculate the probability that the toy lasts less than one year, which is $\int_0^1 f(t)\,dt$. Then, the probability that the toy lasts over one year is $1 - \int_0^1 f(t)\,dt$. Whichever method you use, the answer is $\boxed{\frac{1}{\sqrt{e}}}$.

   (c) *What is the median life of this type of toy?*

Solution. The median is the value T such that $\int_{-\infty}^{T} f(t)\, dt = \frac{1}{2}$. So, we want

$$\begin{aligned} \frac{1}{2} &= \int_{0}^{T} 0.5e^{-0.5t}\, dt \\ &= -e^{-0.5t}\Big|_{0}^{T} \\ &= -e^{-0.5T} + 1 \end{aligned}$$

Solving, $T = \boxed{-2\ln\frac{1}{2}}$ (this is approximately 1.37 years).

2. *A large number of students take an exam. 30% of the students receive a score of 70, 50% receive a score of 80, and 20% receive a score of 90. What is the average score on the exam?*

Solution. If the number of students is N, then $.3N$ people scored 70, $.5N$ scored 80, and $.2N$ scored 90. So, the sum of all scores is $.3N(70) + .5N(80) + .2N(90) = 79N$. The average score is this sum divided by the number of people, or $\boxed{79}$.

3. The density function for the duration of telephone calls within a certain city is $p(x) = 0.4e^{-0.4x}$ where x denotes the duration in minutes of a randomly selected call.

(a) *What percentage of calls last one minute or less?*

Solution. We are interested in the fraction of calls that last between 0 and 1 minute, which is $\int_{0}^{1} 0.4e^{-0.4x}\, dx = -e^{-0.4x}\Big|_{0}^{1} = 1 - e^{-0.4}$. (As a percent, it is $100(1 - e^{-0.4}) \approx 33\%$.)

(b) *What percentage of calls last between one and two minutes?*

Solution. This is $\int_{1}^{2} 0.4e^{-0.4x}\, dx = -e^{-0.4x}\Big|_{1}^{2} = e^{-0.4} - e^{-0.8}$, which is approximately 22%.

(c) *What percentage of calls last 3 minutes or more?*

Solution. This is

$$\begin{aligned} \int_{3}^{\infty} 0.4e^{-0.4x}\, dx &= \lim_{b\to\infty} \int_{3}^{b} 0.4e^{-0.4x}\, dx \\ &= \lim_{b\to\infty} -e^{-0.4x}\Big|_{3}^{b} \\ &= \lim_{b\to\infty} e^{-1.2} - e^{-0.4b} \\ &= e^{-1.2} \end{aligned}$$

This works out to approximately 30%.

(d) *What is the average length of a call?*

Solution. The average length of a call is $\int_{0}^{\infty} x(0.4e^{-0.4x})\, dx$. Using integration by parts, an antiderivative of $x(0.4e^{-0.4x})$ is $-xe^{-0.4x} - 2.5e^{-0.4x}$. So,

$$\begin{aligned} \int_{0}^{\infty} x(0.4e^{-0.4x})\, dx &= \lim_{b\to\infty} \left(-xe^{-0.4x} - 2.5e^{-0.4x}\right)\Big|_{0}^{b} \\ &= \lim_{b\to\infty} -be^{-0.4b} - 2.5e^{-0.4b} + 2.5 \qquad (1) \end{aligned}$$

The middle term, $-2.5e^{-0.4b}$, tends to 0 as $b \to \infty$. For the first term, $-be^{-0.4b}$, we need to use L'Hospital's Rule (since the limit is of the form $\infty \cdot 0$):

$$\begin{aligned}\lim_{b\to\infty} -be^{-0.4b} &= \lim_{b\to\infty} -\frac{b}{e^{0.4b}} \\ &= \lim_{b\to\infty} -\frac{1}{0.4e^{0.4b}} \\ &= 0\end{aligned}$$

Plugging this into (**??**), the average length of a call is $\boxed{\text{2.5 minutes}}$.

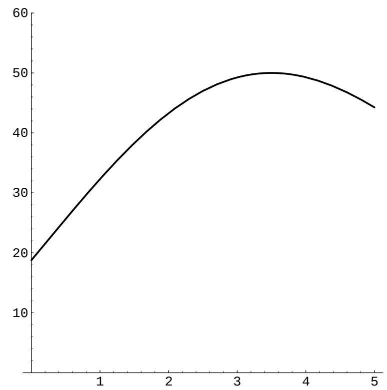
4. *The lifetime, in hundreds of hours, of a certain type of light bulb has been found empirically to have a probability density function approximated by* $f(x) = \frac{\sqrt{65}}{8(1+x^2)^{3/2}}$ *for* $0 < x < 8$. *Find the mean lifetime of such a bulb.*

Solution. The mean lifetime is $\displaystyle\int_0^8 xf(x)\,dx = \frac{\sqrt{65}}{8}\int_0^8 \frac{x}{(1+x^2)^{3/2}}\,dx$. To evaluate this, we substitute $u = 1 + x^2$:

$$\begin{aligned}\frac{\sqrt{65}}{8}\int_0^8 \frac{x}{(1+x^2)^{3/2}}\,dx &= \frac{\sqrt{65}}{8}\int_1^{65} \frac{1}{2}u^{-3/2}\,du \\ &= -\frac{\sqrt{65}}{8}u^{-1/2}\Big|_1^{65} \\ &= \frac{\sqrt{65}}{8}\left(1 - \frac{1}{\sqrt{65}}\right) \\ &= \boxed{\frac{\sqrt{65}-1}{8}}\end{aligned}$$

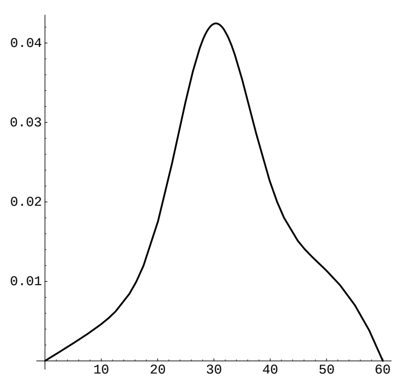
More Probability

1. (a) *The following function represents the temperature outside as a function of time. Estimate the average temperature between time* 0 *and time* 5.



Solution. We are looking for the average value of the temperature function, or the average height of its graph. In this graph, it looks like this is around 40. (The exact answer is around 41.5.)

(b) *A meteorologist takes several temperature readings which are described by the following probability density function. Estimate the average temperature.*



Solution. The probability density function has a peak at 30, but the area under the curve to the left is smaller than the area under the curve to the right. This tells us that more of the readings were above 30 than below 30, so we can guess that the average temperature was a bit above 30. It does not look like the average temperature will be as high as 40 because the fraction of temperature readings above 40 (the area under the graph to the right of 40) is pretty small. So, we guess that the average temperature is above 30 but below 40. (The actual value is about 32.)

2. Let $p(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}$, a probability density function.

(a) *Sketch the graph of this probability density function. What do you think its mean is?*

Solution. It looks like 0 since the graph is symmetric about $x = 0$.

(b) *Verify your guess mathematically.*

Solution. To find the mean, we need to evaluate $\int_{-\infty}^{\infty} x \cdot \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx$. Since the bounds $-\infty$ and ∞ are both improprieties, we need to split the integral into two pieces and evaluate:

$$\int_{-\infty}^{\infty} x \cdot \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx = \frac{1}{\sqrt{2\pi}} \left(\int_{-\infty}^{0} xe^{-x^2/2}\, dx + \int_{0}^{\infty} xe^{-x^2/2}\, dx \right) \tag{1}$$

Let's first find an antiderivative of $xe^{-x^2/2}$. We use the substitution $u = -x^2/2$ to get

$$\int xe^{-x^2/2}\ x = \int -e^u\ du = -e^u + C = -e^{-x^2/2} + C.$$

Now, we'll go back to (**??**). Let's do the integral from 0 to infinity first. Since it's an improper integral, we really need to take a limit:

$$\begin{aligned} \int_0^{\infty} xe^{-x^2/2}\ dx &= \lim_{b\to\infty} \int_0^b xe^{-x^2/2}\ dx \\ &= \lim_{b\to\infty} -e^{-x^2/2}\Big|_0^b \\ &= \lim_{b\to\infty} 1 - e^{-b^2/2} \\ &= 1 \end{aligned}$$

Similarly, we find that $\int_{-\infty}^{0} xe^{-x^2/2}\ dx = -1$, so (**??**) tells us that the mean is $-1 + 1 = \boxed{0}$.

3. *The bell curve with mean* 0 *and standard deviation* s *is given by the probability density function* $p(x) = \frac{1}{s\sqrt{2\pi}} e^{-x^2/(2s^2)}$. *What fraction of the population is within one standard deviation* s *of the mean 0?*

   Solution. We are looking for the fraction of the population that is between $-s$ and s, which is given by the integral $\boxed{\frac{1}{s\sqrt{2\pi}} \int_{-s}^{s} e^{-x^2/(2s^2)}\ dx}$.

1. *Find the degree 6 Taylor polynomial approximation for $f(x) = \sin x$ centered at 0.*

Solution. We are looking for a polynomial $P_6(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6$ such that the k-th derivative $P_6^{(k)}(0)$ is equal to the k-th derivative $f^{(k)}(0)$. The k-th derivative $P_6^{(k)}(0)$ is just equal to $k!a_k$, so we want $a_k = \frac{f^{(k)}(0)}{k!}$. The derivatives of $f(x) = \sin x$ are:

$$\begin{array}{rclcrcl} f(x) &=& \sin x &\Rightarrow& f(0) &=& 0 \\ f'(x) &=& \cos x &\Rightarrow& f'(0) &=& 1 \\ f''(x) &=& -\sin x &\Rightarrow& f''(0) &=& 0 \\ f'''(x) &=& -\cos x &\Rightarrow& f'''(0) &=& -1 \\ f^{(4)}(x) &=& \sin x &\Rightarrow& f^{(4)}(0) &=& 0 \\ f^{(5)}(x) &=& \cos x &\Rightarrow& f^{(5)}(0) &=& 1 \\ f^{(6)}(x) &=& -\sin x &\Rightarrow& f^{(6)}(0) &=& 0 \end{array}$$

Therefore, $a_0 = 0$, $a_1 = 1$, $a_2 = 0$, $a_3 = -\frac{1}{3!}$, $a_4 = 0$, $a_5 = \frac{1}{5!}$, and $a_6 = 0$. So, $P_6(x) = \boxed{x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5}$.

2. (a) *If you want to find a Taylor polynomial approximation $a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$ (centered at 0) to $f(x)$, write a formula for the coefficient a_k.*

   Solution. $\boxed{a_k = \frac{f^{(k)}(0)}{k!}}$. (You might wonder what happens when $k = 0$: $0!$ is defined to be 1, so the formula still works.)

   (b) *If you want to find a Taylor polynomial approximation $a_0 + a_1(x-3) + a_2(x-3)^2 + \cdots + a_n(x-3)^n$ (centered at 3) to $f(x)$, write a formula for the coefficient a_k.*

   Solution. $\boxed{a_k = \frac{f^{(k)}(3)}{k!}.}$

3. *How do you think you would represent $\sin x$ as an infinite polynomial centered at 0? This is called the Taylor series (rather than Taylor polynomial) generated by $\sin x$ about 0.*

Solution. Based on the pattern we started to see in #1, it seems like we should get

$$\boxed{x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \frac{1}{9!}x^9 - \cdots}.$$

4. *What is the Taylor series generated by $\cos x$ about 0?*

Solution. Using $f(x) = \cos x$, we have

$$\begin{array}{rclcrcl} f(x) &=& \cos x &\Rightarrow& f(0) &=& 1 \\ f'(x) &=& -\sin x &\Rightarrow& f'(0) &=& 0 \\ f''(x) &=& -\cos x &\Rightarrow& f''(0) &=& -1 \\ f'''(x) &=& \sin x &\Rightarrow& f'''(0) &=& 0 \end{array}$$

After this, the derivatives repeat, and we continue to get $1, 0, -1, 0, 1, 0, -1, 0, \ldots$. So, our coefficients are $a_0 = 1$, $a_1 = 0$, $a_2 = -\frac{1}{2}!$, $a_3 = 0$, $a_4 = \frac{1}{4!}$, $a_5 = 0$, and so on. Thus, the Taylor series should be

$$\boxed{1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 - \cdots}.$$

5. *What is the Taylor series generated by e^x about 0?*

   Solution. If $f(x) = e^x$, then the k-th derivative $f^{(k)}(x)$ is always e^x, so $f^{(k)}(0) = 1$. Therefore, the k-th coefficient a_k in the Taylor series is $\frac{1}{k!}$. Thus, the Taylor series is $\boxed{1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots}$.

   If we wanted to write this in summation notation, we would write $\displaystyle\sum_{k=0}^{\infty} \frac{x^k}{k!}$.

6. We hope that, by using "polynomials of infinite degree," we end up with something that is not just an approximation for our function but is actually equal to the function. We don't really know if this is true yet. Taking on faith that e^x is actually equal to its Taylor expansion about 0, can you write a power series expansion (or "infinite polynomial representation") of:

   (a) e^{-x^2}?

   Solution. Since $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$, we can get e^{-x^2} just by replacing all of the x's in the series for e^x with $-x^2$: $e^{-x^2} = \boxed{1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \frac{x^8}{4!} - \cdots}$. In summation notation, $e^{-x^2} = \displaystyle\sum_{k=0}^{\infty} \frac{(-x^2)^k}{k!}$. We often simplify $(-x^2)^k$ as $[(-1)(x^2)]^k = (-1)^k x^{2k}$, so you might also see this as $\displaystyle\sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{k!}$.

   (b) $\displaystyle\int e^{-x^2}\, dx$?

   Solution. If we believe that $e^{-x^2} = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \frac{x^8}{4!} - \cdots$, then it seems plausible that we can integrate this using the reverse of the Power Rule to get $\displaystyle\int e^{-x^2}\, dx = C + x - \frac{1}{3}x^3 + \frac{1}{5} \cdot \frac{x^5}{2!} - \frac{1}{7} \cdot \frac{x^7}{3!} + \frac{1}{9} \cdot \frac{x^9}{4!} - \cdots$, where C is any constant.

7. (a) *Write a general formula for the Taylor series of $f(x)$ centered at 0.*

   Solution. We know it should look like $a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$ where $a_k = \frac{f^{(k)}(0)}{k!}$. So, it is $\boxed{f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots}$. In summation notation, it is $\displaystyle\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k$.

   (b) *What if you wanted to center at 5?*

   Solution. Then, we would get $a_0 + a_1(x-5) + a_2(x-5)^2 + a_3(x-5)^3 + \cdots$ where $a_k = \frac{f^{(k)}(5)}{k!}$. In other words, we would get $\boxed{f(5) + f'(5)(x-5) + \frac{f''(5)}{2!}(x-5)^2 + \frac{f'''(5)}{3!}(x-5)^3 + \cdots}$.

Geometric Sums and Geometric Series

1. In your quest to become a millionaire by age 50, you start an aggressive savings plan. You open a new investment account on January 1, 2008 and deposit \$9000 into it every year on January 1. Each year, you earn 7% interest on December 31.

   (a) *How much money will you have in your account on January 2, 2009? 2010? 2014? (Don't try to add or multiply things out; just write an arithmetic expression.)*

   Solution. On December 31, 2008, you will have \$9000(1.07) because of the interest you've earned. After you deposit \$9000 on January 1, 2009, you will have $\$9000 + \$9000(1.07)$.

   On December 31, 2009, you will receive your interest, giving you $\$9000(1.07)+\$9000(1.07)^2$. After you make your yearly deposit on January 1, 2010, you will have $\$9000+\$9000(1.07)+\$9000(1.07)^2$.

   Continuing this reasoning, you will have $\$9000 + \$9000(1.07) + \$9000(1.07)^2 + \cdots + \$9000(1.07)^6$ on January 2, 2014.

   (b) *Will you be a millionaire by age 50?*

   Solution. The answer will depend on when you were born. Let's say that you were born in 1988. Then, we want to know what has happened by the year 2038. Using the argument of part (a), on January 2, 2038, you will have $\$9000 + \$9000(1.07) + \$9000(1.07)^2 + \cdots + \$9000(1.07)^{30}$. To see whether this is bigger than a million, we want some way to calculate this sum quickly.

   Let's call this amount S:

   $$S = 9000 + 9000(1.07) + 9000(1.07)^2 + 9000(1.07)^3 + \cdots + 9000(1.07)^{29} + 9000(1.07)^{30} \qquad (1)$$

   Notice that if we multiply both sides by 1.07, we get something similar looking on the right side:

   $$1.07S = 9000(1.07) + 9000(1.07)^2 + 9000(1.07)^3 + \cdots + 9000(1.07)^{30} + 9000(1.07)^{31} \qquad (2)$$

   Subtracting (**??**) from (**??**), most of the terms on the right side cancel, and we are left with $0.07S = 9000(1.07)^{31} - 9000$, so $S = \frac{9000(1.07)^{31}-9000}{.07} = \$918,657.37$. So, you are not quite a millionaire, but you are close!

2. If you suffer from allergies, your doctor may suggest that you take Claritin once a day. Each Claritin tablet contains 10 mg of loratadine (the active ingredient). Every 24 hours, about 7/8 of the loratadine in the body is eliminated (so 1/8 remains). [1]

   (a) *If you take one Claritin tablet every morning for a week, how much loratadine is in your body right after you take the 3rd tablet? 7th tablet? (Don't try to simplify your computations; just write out an arithmetic expression.)*

   Solution. Immediately after taking the first tablet, you have 10 mg of loratadine in your body. The following morning, only 1/8 of that is left, so you have 10 (1/8) mg in your body. You then take another pill containing 10 mg, so you have a total of $10 + 10(1/8)$ mg of loratadine in your body.

   The following morning, 1/8 of that remains, or $[10 + 10(1/8)](1/8) = 10(1/8) + 10(1/8)^2$. You then take another pill containing 10 mg, so you have a total of $10 + 10(1/8) + 10(1/8)^2$ mg of loratadine in your body after taking the 3rd pill.

[1] This estimate comes from the fact that the average half-life of loratadine is known to be 8 hours.

Continuing this reasoning, you will have $10+10(1/8)+10(1/8)^2+10(1/8)^3+10(1/8)^4+10(1/8)^5+10(1/8)^6$ mg in your body after the 7th pill.

(b) *If you take Claritin for years and years, will the amount of loratadine in your body level off? Or will your bloodstream be pure loratadine?*

Solution. Right after you take the n-th pill, you will have $10+10(1/8)+10(1/8)^2+10(1/8)^3+\cdots+10(1/8)^{n-1}$ mg of loratadine in your body. Let's call this amount S_n. We are wondering what happens to S_n as n gets very large.

We use the same trick we used in #1(b) to get a closed form expression for S_n. We said that

$$S_n = 10+10(1/8)+10(1/8)^2+\cdots+10(1/8)^{n-2}+10(1/8)^{n-1} \tag{3}$$

Multiplying both sides by 1/8, we get that

$$(1/8)S_n = 10(1/8)+10(1/8)^2+10(1/8)^3+\cdots+10(1/8)^{n-1}+10(1/8)^n \tag{4}$$

If we subtract (**??**) from (**??**), most of the terms on the right side cancel, and we are left with $(7/8)S_n = 10-10(1/8)^n$. Dividing both sides by 7/8, $S_n = \frac{10-10(1/8)^n}{7/8}$.

Using this expression for S_n, it is easy to see what happens as n gets bigger and bigger: $(1/8)^n$ gets closer and closer to 0, so $\frac{10-10(1/8)^n}{7/8}$ gets closer and closer to $\frac{10}{7/8}=\frac{80}{7}$. Thus, over time, the amount of loratadine in your body gets closer and closer to $\frac{80}{7}$ mg.

3. *For what values of x does the geometric series $1+x+x^2+\cdots$ converge?* [2] *If it converges, what does it converge to?*

Solution. This is a geometric series $a+ar+ar^2+ar^3+\cdots$ with $a=1$ and $r=x$. Therefore, we know that it diverges when $|x|\geq 1$. When $|x|<1$, it converges to $\frac{1}{1-x}$.

4. Which of the following series are geometric?

(a) $\displaystyle\sum_{k=1}^{\infty}\frac{(-1)^k 2^{k+1}}{3^k}$.

Solution. A geometric series is a series of the form $a+ar+ar^2+\cdots$. To decide whether the given series is geometric, we want to see if it matches this form. It's helpful to write out the first few terms. When $k=1$, we have $\frac{(-1)^1 2^2}{3^1}=-\frac{4}{3}$. When $k=2$, we have $\frac{(-1)^2 2^3}{3^2}=\frac{8}{9}$. When $k=3$, we have $\frac{(-1)^3 2^4}{3^3}=-\frac{16}{27}$. When $k=4$, we have $\frac{(-1)^4 2^5}{3^4}=\frac{32}{81}$. So far, it looks like it could be the geometric series $a+ar+ar^2+ar^3+\cdots$ with $a=-\frac{4}{3}$ and $r=-\frac{2}{3}$.

To see if this is correct, let's compare the k-th terms. The k-th term of the given series is $\frac{(-1)^k 2^{k+1}}{3^k}$, while the k-th term of the geometric series is ar^{k-1}. So, we are hoping that $\frac{(-1)^k 2^{k+1}}{3^k} = \left(-\frac{4}{3}\right)\left(-\frac{2}{3}\right)^{k-1}$. If you multiply out the right side, you will see that this is indeed the case, so the given series is geometric with $a=-\frac{4}{3}$ and $r=-\frac{2}{3}$.

(b) $\displaystyle\sum_{k=1}^{\infty}\frac{1}{k^3}$.

[2] We could also write this series in summation notation as $\displaystyle\sum_{k=0}^{\infty}x^k$.

Solution. If we write this series out, it is $1 + \frac{1}{8} + \frac{1}{27} + \frac{1}{64} + \cdots$. We can already see that it's not geometric because the terms don't have a common ratio. (To elaborate: if it was geometric, the first term would be a and the second term would be ar; this means that a would have to be 1, and r would have to be $\frac{1}{8}$. But then the third term isn't right.)

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{2}{3^{n/2}}$.

Solution. We could also write this series as $\frac{2}{3^{1/2}} + \frac{2}{3} + \frac{2}{3^{3/2}} + \frac{2}{3^2} + \cdots$, which looks like it might be geometric with $a = \frac{2}{3^{1/2}}$ and $r = \frac{1}{3^{1/2}}$.

To check if this is correct, we want to see if the n-th term is $ar^{n-1} = \frac{1}{3^{1/2}} \left(\frac{1}{3^{1/2}}\right)^{n-1}$, and it is. So, the series is geometric with $a = \frac{2}{3^{1/2}}$ and $r = \frac{1}{3^{1/2}}$.

1. Suppose you know that the infinite series $a_1 + a_2 + a_3 + \cdots + a_n + \cdots$ converges to s and that $a_k > 0$ for k any positive integer. Let $s_n = a_1 + a_2 + a_3 + \cdots + a_n$. For each of the following statements, determine whether the statement must be true, could possibly be true, or must be false.

   (a) $\lim_{n\to\infty} a_n = 0$.

   (b) $\lim_{n\to\infty} s_n = 0$.

   (c) There exists a number M such that $s_n < M$ for all n. (This is equivalent to saying that the partial sums are bounded. Why?)

   (d) $\sum_{k=5}^{\infty} a_k$ converges.

   Solution. (a) must be true, (b) must be false, (c) must be true, and (d) must be true. (See the solutions to Homework 15 for more details.)

2. *Suppose you know that* $\lim_{n\to\infty} b_n = 0$. *Can you be sure that the infinite series* $b_1+b_2+b_3+\cdots$ *converges?*

   Solution. No; the harmonic series in #5 is an example of a series that diverges even though its terms tend to 0.

3. (a) *Give an example of a sequence (ordered list) of numbers such that the numbers are increasing but are bounded.*

   Solution. $0.9, 0.99, 0.999, 0.9999, 0.99999, \ldots$ is one such example.

   (b) *Give an example of a sequence (ordered list) of numbers such that the numbers are increasing and are not bounded.*

   Solution. $1, 2, 3, 4, 5, \ldots$.

   (c) *Give an example of a sequence (ordered list) of numbers such that the numbers are bounded but have no limit as* $n \to \infty$.

   Solution. $0, 1, 0, 1, 0, 1, 0, 1, \ldots$.

4. (a) *A sequence which is both monotonic and bounded*

   must converge   *could either converge or diverge*   *must diverge*

   Solution. Must converge. This is the Monotonic Sequence Theorem.

   (b) *A sequence which is monotonic but not bounded*

   must converge   *could either converge or diverge*   *must diverge*

   Solution. Must diverge.

5. Consider the series $\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$ (called the harmonic series).

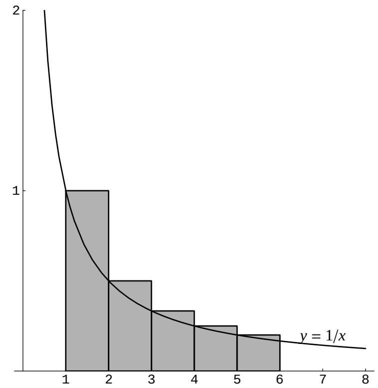
(a) *Does the sequence of terms converge? If so, to what does it converge?*

Solution. Yes, the terms $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$ are getting closer to 0.

(b) *Does the sequence of partial sums converge? If so, to what does it converge?*

Solution. The sequence of partial sums does not converge. The sequence of partial sums is definitely increasing (every partial sum is bigger than the previous partial sum). So, if we knew the sequence of partial sums was not bounded, we could conclude that it didn't converge (by #4(b)).

One way to see that the sequence of partial sums is not bounded is to look at them graphically. The partial sums can be represented as areas. For instance, the 5th partial sum is $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}$, which we can represent as the area in these 5 boxes:



As we can see, the 5th partial sum is therefore bigger than $\int_1^6 \frac{1}{x}\,dx$. In general, the n-th partial sum is bigger than $\int_1^{n+1} \frac{1}{x}\,dx$. Taking the limit as $n \to \infty$ of $\int_1^{n+1} \frac{1}{x}\,dx$ gives us the improper integral $\int_1^{\infty} \frac{1}{x}\,dx$, which we know diverges. So, the sequence of partial sums also diverges.

The other way to understand these partial sums is using a clever trick. Let's look at the partial sums $s_1, s_2, s_4, s_8, s_{16}, \ldots$.

$$
\begin{aligned}
s_1 &= 1 \\
s_2 &= 1+\frac{1}{2} \\
s_4 &= 1+\frac{1}{2}+\left(\frac{1}{3}+\frac{1}{4}\right) \\
&> 1+\frac{1}{2}+\left(\frac{1}{4}+\frac{1}{4}\right) \\
&= 1+\frac{1}{2}+\frac{1}{2} \\
s_8 &= 1+\frac{1}{2}+\left(\frac{1}{3}+\frac{1}{4}\right)+\left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right) \\
&> 1+\frac{1}{2}+\left(\frac{1}{4}+\frac{1}{4}\right)+\left(\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}\right) \\
&= 1+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}
\end{aligned}
$$

$$
\begin{aligned}
s_{16} &= 1+\frac{1}{2}+\left(\frac{1}{3}+\frac{1}{4}\right)+\left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right)+\left(\frac{1}{9}+\frac{1}{10}+\cdots+\frac{1}{16}\right) \\
&> 1+\frac{1}{2}+\left(\frac{1}{4}+\frac{1}{4}\right)+\left(\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}\right)+\left(\frac{1}{16}+\frac{1}{16}+\cdots+\frac{1}{16}\right) \\
&= 1+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}
\end{aligned}
$$

Since this pattern continues, there is no way that the sequence of partial sums can be bounded.

(c) *Does the series* $\displaystyle\sum_{k=1}^{\infty}\frac{1}{k}$ *converge?*

Solution. This is really the same question as (b); the sequence of partial sums diverges, so the series diverges.

(d) *Does the series* $\displaystyle\sum_{k=10^{10}}^{\infty}\frac{1}{100000000k}$ *converge?*

Solution. No. We know that $\displaystyle\sum_{k=1}^{\infty}\frac{1}{k}$ diverges. Therefore, the series $\displaystyle\sum_{k=1}^{\infty}\frac{1}{100000000k}$ diverges (the whole thing has just been multiplied by a constant). We know that the beginning terms of a series do not affect its convergence, so if we start with the 10^{10}-th term, the series still diverges.

6. See if you can determine whether each of the following series converges or diverges by using the Nth Term Test for Divergence, results about geometric series, or some sort of comparison to series you know about.

(a) $\displaystyle\sum_{k=100}^{\infty}\frac{1}{3k}$.

Solution. This is basically the harmonic series multiplied by $\frac{1}{3}$, except that the first 99 terms are missing. Neither of things affects convergence, so this series $\boxed{\text{diverges}}$, just like the harmonic series.

(b) $\displaystyle\sum_{k=3}^{\infty}\frac{(-1)^k 2^k}{3^k}$.

Solution. We can rewrite this as $\displaystyle\sum_{k=3}^{\infty}\left(-\frac{2}{3}\right)^k$, which is a geometric series with common ratio $-\frac{2}{3}$. We know that such a series $\boxed{\text{converges}}$.

(c) $\displaystyle\sum_{k=4}^{\infty}\frac{(-1)^k 3^k}{2^k}$.

Solution. We can rewrite this as $\displaystyle\sum_{k=4}^{\infty}\left(-\frac{3}{2}\right)^k$, which is a geometric series with common ratio $-\frac{3}{2}$. We know that such a series $\boxed{\text{diverges}}$. Alternatively, you could use the Nth Term Test for Divergence: $\displaystyle\lim_{k\to\infty}\frac{(-1)^k 3^k}{2^k}$ does not exist, so the Nth Term Test for Divergence tells us that the series cannot converge either.

(d) $\sum_{n=1}^{\infty} \frac{\ln n}{n}$.

Solution. Intuitively, we know that a series converges if its terms go to 0 "quickly enough." The terms of this series do go to 0, but $\frac{\ln n}{n} > \frac{1}{n}$ when $n > e$, so the terms of the given series go to 0 more slowly than the terms of the harmonic series. Since we know that the harmonic series diverges, we should guess that this series diverges also.

To make it more precise mathematically, we could say that the partial sums of this series grow more quickly than the partial sums of the harmonic series. Since the partial sums of the harmonic series already grow without bound, the partial sums of this series also grow without bound, so this series $\boxed{\text{diverges}}$.

(e) $\sum_{n=2}^{\infty} \frac{n}{\ln n}$.

Solution. We can use the Nth Term Test for Divergence: $\lim_{n\to\infty} \frac{n}{\ln n} = \lim_{n\to\infty} \frac{1}{1/n} = \lim_{n\to\infty} n = \infty$ (using L'Hospital's Rule in the first step). Since the terms of this series do not tend to 0, the series $\boxed{\text{diverges}}$.

(f) $\sum_{n=0}^{\infty} \sin n$.

Solution. Again, we can use the Nth Term Test for Divergence: $\lim_{n\to\infty} \sin n$ does not exist, so the series $\boxed{\text{diverges}}$.

Comparison

1. Use the Comparison Test (also known as "direct comparison") to decide whether the following series converge or diverge.

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}3^n}$.

   Solution. When n is really big, 3^n is much much bigger than $\sqrt{n}$, so it seems like the terms of this series are affected more by 3^n than $\sqrt{n}$. Therefore, let's compare to the geometric series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3^n}$. When $n \geq 1$, $0 \leq \frac{1}{\sqrt{n}3^n} \leq \frac{1}{3^n}$. The geometric series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3^n}$ converges (the common ratio is 1/3), so the Comparison Test tells us that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}3^n}$ also converges.

   (b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n!}$.

   Solution. If we "unpack" the summation notation, we get $1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} + \cdots$. These terms seem to go to 0 pretty quickly — certainly more quickly, say, than the geometric series $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$, which we know converges. So, let's compare to the geometric series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^{n-1}}$. (Why is it 2^{n-1} instead of 2^n, you might be wondering? Because we're starting with $n = 1$, but we want the first term in the geometric series to be 1.)

   We'd like to say that $0 \leq \frac{1}{n!} \leq \frac{1}{2^{n-1}}$. The first inequality is obviously true. Now, $\frac{1}{n!}$ means $\frac{1}{n} \cdot \frac{1}{n-1} \cdot \frac{1}{n-2} \cdots \frac{1}{2}\frac{1}{1} = \frac{1}{n} \cdot \frac{1}{n-1} \cdot \frac{1}{n-2} \cdots \frac{1}{2}$, while $\frac{1}{2^{n-1}}$ means $\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdots \frac{1}{2}$. Both products have $n-1$ terms, and all of the terms in the product for $\frac{1}{2^{n-1}}$ are at least as big as the corresponding term in $\frac{1}{n!}$. So, it is indeed the case that $0 \leq \frac{1}{n!} \leq \frac{1}{2^{n-1}}$. Since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^{n-1}}$ converges (it's a geometric series with common ratio 1/2), the Comparison Test tells us that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n!}$ also converges.

   (c) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{n^3 - n + 1000}$.

   Solution. You might have the intuition that the terms $\frac{n^2}{n^3-n+1000}$ "grow like" $\frac{n^2}{n^3} = \frac{1}{n}$, so the series should act like $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ and diverge. To verify this, we should compare to $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$.

   Let's try: we want to say that $0 \leq \frac{1}{n} \leq \frac{n^2}{n^3-n+1000}$. The first inequality is always true, but the second inequality is only true when $n^3 \geq n^3 - n + 1000$, or $n \geq 1000$. This isn't a big problem though; as we know, the beginning terms of a series don't affect convergence.

   So, here's the appropriate reasoning: $0 \leq \frac{1}{n} \leq \frac{n^2}{n^3-n+1000}$ when $n \geq 1000$. We know that $\displaystyle\sum_{n=1000}^{\infty} \frac{1}{n}$ diverges (this is the harmonic series without the first 999 terms, and we know the harmonic series diverges). Therefore, the Comparison Test tells us that $\sum_{n=1000}^{\infty} \frac{n^2}{n^3 - n + 1000}$ also diverges.

Adding in finitely many terms at the beginning doesn't affect convergence, so $\sum_{n=1}^{\infty} \frac{n^2}{n^3 - n + 1000}$ also diverges.

(d) $\sum_{n=1}^{\infty} \frac{1}{\ln(1+n)}$.

Solution. The only series we understand so far are geometric series and the harmonic series. We know that $\ln x$ grows more slowly than x, so it seems like we should compare this to the harmonic series.

By graphing $\ln(1+x)$ and x, we can see that $\ln(1+x) \leq x$ when $x \geq 0$, so $\frac{1}{\ln(1+n)} \geq \frac{1}{n} \geq 0$ for $n \geq 1$. Therefore, the Comparison Test tells us that $\sum_{n=1}^{\infty} \frac{1}{\ln(1+n)}$ diverges.

(To prove that $\ln(1+x) \leq x$, you could do something like this: using what you learned in Math 1a, you can show that the global minimum of $x - \ln(1+x)$ is 0. This means that $x - \ln(1+x) \geq 0$ for all x, so $x \geq \ln(1+x)$.)

2. *True or false: If $\{a_n\}$ is a sequence with positive terms and $\lim_{n\to\infty} a_n = 0$, then there is a number k such that $a_n < 1$ whenever $n \geq k$.*

Solution. Roughly, the statement is saying that, if $\{a_n\}$ is a sequence of positive numbers whose limit is 0, then after a while, all of the numbers in the sequence must be less than 1. This is true; after all, if the limit is 0, that means all of the terms in the sequence must stay really close to 0 after a while.

3. Decide whether the following series converge or diverge using any method you like.

(a) $\sum_{n=100}^{\infty} \cos n$.

Solution. We know that a series converges if its terms go to 0 "quickly enough". In this case, the terms aren't going to 0 at all!

To make this precise, $\lim_{n\to\infty} \cos n$ does not exist, so the Nth Term Test for Divergence says that the series will diverge.

(b) $\sum_{k=1}^{\infty} \frac{(-1)^k 2^{k+1}}{3^k}$.

Solution. We can rewrite the k-th term as $\frac{(-1)^k \cdot 2 \cdot 2^k}{3^k} = 2\left(\frac{-2}{3}\right)^k$, so the series is $\sum_{k=1}^{\infty} 2\left(\frac{-2}{3}\right)^k$. This is a geometric series whose first term is $-\frac{4}{3}$ and whose common ratio is $-\frac{2}{3}$. Therefore, we know that this series converges.

(c) $1 + 0 + (-1) + 1 + 0 + (-1) + 1 + 0 + (-1) + \cdots$.

Solution. One way to approach this is simply to write down the sequence of partial sums: $1, 1, 0, 1, 1, 0, 1, 1, 0, \ldots$. Since the sequence of partial sums diverges, the series diverges. (This is using the *definition* of convergence/divergence of a series, not any particular test.)

Alternatively, we could use the Nth Term Test for Divergence: the sequence of terms is $a_1 = 1, a_2 = 0, a_3 = -1, a_4 = 1, a_5 = 0, a_6 = -1, \ldots$. Since $\lim_{n\to\infty} a_n$ does not exist, the Nth Term Test for Divergence says that the series must diverge.

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n}$.

Solution. We can compare this to the harmonic series: $\frac{\ln n}{n} \geq \frac{1}{n} \geq 0$ as long as $n \geq 3$. The series $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n}$ diverges, so the Comparison Test tells us that $\displaystyle\sum_{n=3}^{\infty} \frac{\ln n}{n}$ diverges as well. Adding on finitely many terms at the beginning doesn't change whether a series converges or diverges, so $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n}$ diverges.

More Comparison

1. *For what values of p is the* <u>*p-series*</u> $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ *convergent?*

   Solution. If we let $f(x) = \frac{1}{x^p}$, then the terms of our series are just $f(1), f(2), f(3), \ldots$. Also, $f(x)$ is positive, continuous, and decreasing on $[1, \infty)$, so we can use the Integral Test. The Integral Test says $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ and $\displaystyle\int_1^{\infty} \frac{1}{x^p}\,dx$ either both converge or both diverge.

   We know that $\displaystyle\int_1^{\infty} \frac{1}{x^p}$ converges when $p > 1$ and diverges when $p \leq 1$. So, the p-series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges when $p > 1$ and diverges when $p \leq 1$.

2. *Does the series* $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^2 - 1}$ *converge or diverge?*

   Solution. This looks quite similar to $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^2}$, but direct comparison doesn't work too well because $\frac{1}{n^2-1} \geq \frac{1}{n^2}$. Instead, we'll use the Limit Comparison Test and compare to $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^2}$.

   We have $\displaystyle\lim_{n\to\infty} \frac{1/(n^2-1)}{1/n^2} = \lim_{n\to\infty} \frac{n^2}{n^2-1} = 1$. Since $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^2}$ converges (by #1), the Limit Comparison Test says that $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^2 - 1}$ converges.

3. *Does the series* $\displaystyle\sum_{k=1}^{\infty} \frac{3}{8^k - 2}$ *converge or diverge?*

   Solution. When k is large, $\frac{3}{8^k-2} \approx \frac{3}{8^k}$. The series $\displaystyle\sum_{k=1}^{\infty} \frac{3}{8^k}$ is geometric with common ratio $\frac{1}{8}$, so it converges. Therefore, we expect our series to converge as well. However, we can't compare directly because $\frac{3}{8^k-2} \geq \frac{3}{8^k}$. Instead, we'll use the Limit Comparison Test to compare these two series.

   We have $\displaystyle\lim_{k\to\infty} \frac{3/(8^k-2)}{3/8^k} = \lim_{k\to\infty} \frac{8^k}{8^k-2} = 1$. Since $\displaystyle\sum_{k=1}^{\infty} \frac{3}{8^k}$ converges, the Limit Comparison Test says that $\displaystyle\sum_{k=1}^{\infty} \frac{3}{8^k - 2}$ converges as well.

4. *Otto is given the following problem for homework.*

   > *Decide whether the series* $\displaystyle\sum_{n=1}^{\infty} \sin^2(\pi n)$ *converges or diverges. Explain your reasoning.*

   Otto writes

The improper integral $\int_1^\infty \sin^2(\pi x)\,dx$ *diverges, so* $\sum_{n=1}^{\infty} \sin^2(\pi n)$ *also diverges by the Integral Test.*

Otto is correct that the improper integral diverges (although he should have shown more work!). But the rest of his reasoning is incorrect — why? And what is the correct answer to the problem?

Solution. The Integral Test doesn't apply to Otto's problem because the function $\sin^2(\pi x)$ is not a decreasing function. In fact, using the Integral Test gives Otto the wrong answer: $\sin^2(\pi n) = 0$ whenever n is an integer, so the series in Otto's problem is just $0 + 0 + 0 + \cdots$, which converges.

5. Let $\sum a_n$ and $\sum b_n$ be series with positive terms. The Limit Comparison Test only applies when $\lim_{n\to\infty} \frac{a_n}{b_n}$ is a positive real number.

   (a) *If* $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$*, can you draw any conclusions?*

   Solution. Intuitively, the fact that $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ means that a_n goes to 0 a lot more quickly than b_n. So, if $\sum b_n$ converges, then $\sum a_n$ converges.

   In fact, you showed this in Problem Set 17 (#38a from Stewart §8.3).

   If $\sum b_n$ diverges, $\sum a_n$ could still converge; an example is $a_n = \frac{1}{n}$ and $b_n = 1$.

   (b) *If* $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$*, can you draw any conclusions?*

   Solution. Intuitively, the fact that $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$ means that a_n goes to 0 a lot more slowly than b_n. So, if $\sum b_n$ diverges, then $\sum a_n$ diverges as well.

   To make this mathematically correct, we can use the Comparison Test. The fact that $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$ means that, eventually (when n is big enough), $\frac{a_n}{b_n}$ will always be greater than 1. So, eventually, $a_n > b_n$, and the Comparison Test says that $\sum a_n$ will have to diverge since $\sum b_n$ does. (Just like in #1(c) from the "Comparison" handout, it's not true that $a_n > b_n$ for all n, just when n is big enough. But that's fine because we know the beginning terms of a series don't affect whether it converges.)

   If $\sum b_n$ converges, we can't draw any conclusion about $\sum a_n$.

1. *Does the series* $\sum_{k=1}^{\infty}(-1)^{k+1}\frac{1}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots$ *converge or diverge? (This series is often called the alternating harmonic series.)*

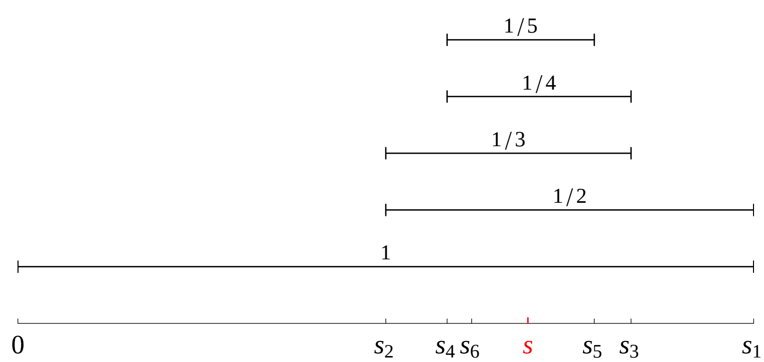
   Solution. To see if the series converges, let's see if we can apply the Alternating Series Test. The terms are decreasing in magnitude: $\frac{1}{k+1} \leq \frac{1}{k}$. In addition, the terms approach 0: $\lim_{k\to\infty}\frac{1}{k} = 0$. Therefore, the Alternating Series Test applies, and we can conclude that $\sum_{k=1}^{\infty}(-1)^{k+1}\frac{1}{k}$ converges.

2. *In fact,* $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots = \ln 2$. *Write a finite sum which estimates* $\ln 2$ *with error of less than* 0.001. *Is your approximation too big or too small?*

   Solution. The Alternating Series Estimate tells us that the magnitude of the error in using the n-th partial sum $s_n = \sum_{k=1}^{n}(-1)^{k+1}\frac{1}{k}$ is at most the magnitude of the next term, or $\frac{1}{n+1}$. That is, $|s - s_n| \leq \frac{1}{n+1}$. We want $|s - s_n| < 0.001$, so let's find an n satisfying $\frac{1}{n+1} < 0.001$. (Then we'll have $|s - s_n| \leq \frac{1}{n+1} < 0.001$.)

   To get $\frac{1}{n+1} < 0.001$, we need $n + 1 > 1000$. The smallest n which makes this work is $n = 1000$, so we can use the 1000th partial sum $\boxed{1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots - \frac{1}{999} + \frac{1}{1000}}$.

   To determine whether our approximation is too big or too small, let's go back and look at our diagram of partial sums. Here it is, with the actual sum s in red:



   From the diagram, we can see that the even partial sums $s_2, s_4, s_6, \ldots$ are all $\boxed{\text{too small}}$.

3. *Is the alternating harmonic series* $\sum_{k=1}^{\infty}(-1)^{k+1}\frac{1}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots$ *absolutely convergent?*

Solution. Asking whether the series is absolutely convergent is the same as asking whether the series $\sum_{n=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots$ converges. This is the harmonic series, which diverges. Thus, the series $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k}$ is not absolutely convergent.

4. Determine whether each series converges or diverges. If it converges, does it converge absolutely or conditionally?

   (a) $\sum_{k=1}^{\infty} \frac{(-1)^k}{\sqrt{k}}$.

   Solution. We can apply the Alternating Series Test: $\frac{1}{\sqrt{k+1}} \leq \frac{1}{\sqrt{k}}$ and $\lim_{k\to\infty} \frac{1}{\sqrt{k}} = 0$, so the Alternating Series Test says that $\sum_{k=1}^{\infty} \frac{(-1)^k}{\sqrt{k}}$ converges.

   To decide whether it converges absolutely, we look at the series of absolute values, $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$. This is a p-series with $p = \frac{1}{2}$, and we know that diverges. So, $\sum_{k=1}^{\infty} \frac{(-1)^k}{\sqrt{k}}$ $\boxed{\text{converges conditionally}}$.

   (b) $\sum_{k=1}^{\infty} \frac{\sin k}{k!}$.

   Solution. This series is not an alternating series, so we should not even try the Alternating Series Test. All of our other tests involve series with positive terms, so let's look at the series of absolute values first. That is, we'll first look at $\sum_{k=1}^{\infty} \frac{|\sin k|}{k!}$.

   We can use the Comparison Test here: $0 \leq \frac{|\sin k|}{k!} \leq \frac{1}{k!}$. We know that $\sum_{k=1}^{\infty} \frac{1}{k!}$ converges (see #1(b) from the "Comparison" handout), so $\sum_{k=1}^{\infty} \frac{|\sin k|}{k!}$ converges.

   This means that $\sum_{k=1}^{\infty} \frac{\sin k}{k!}$ $\boxed{\text{converges absolutely}}$. We know that a series which converges absolutely also converges.

   (c) $\sum_{k=1}^{\infty} (-2)^k$.

   Solution. This is an alternating series, but the Alternating Series Test does not apply because the terms are not decreasing in magnitude. That does tell us whether the series converges though.

   Instead, notice that the terms aren't going to 0: $\lim_{k\to\infty} (-2)^k$ does not exist. So, the Nth Term Test for Divergence says that the series $\boxed{\text{diverges}}$. (Alternatively, you could justify by saying that the series is geometric with common ratio -2.)

5. (a) *The Taylor series for* $\cos x$ *about 0 is* $1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{k=0}^{\infty}(-1)^{k+1}\frac{x^{2k}}{(2k)!}$. *Show that, if you plug in any value of* x *with* $-0.5 \leq x \leq 0.5$, *the series converges.*

Solution. We will show that we can apply the Alternating Series Test. If $-0.5 \leq x \leq 0.5$, then $\frac{x^{2(k+1)}}{[2(k+1)]!} \leq \frac{x^{2k}}{(2k)!}$. This is true because the expression on the left has a bigger denominator and smaller numerator than the expression on the right (in the numerator, x is raised to a higher exponent, and $|x| < 1$). So, the first condition in the Alternating Series Test is satisfied.

Next, we need to show that $\lim_{k\to\infty} \frac{x^{2k}}{(2k)!} = 0$. In fact, $\lim_{k\to\infty} x^{2k} = 0$ (since x is between -0.5 and 0.5), so $\lim_{k\to\infty} \frac{x^{2k}}{(2k)!} = 0$ as well.

Therefore, the Alternating Series Test applies, and we can conclude that the series converges.

(b) *In fact, the series converges for all* x, *and* $\cos x$ *is actually equal to the series; that is,*

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots.$$

Suppose you use the approximation $\cos x \approx 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$ *to approximate* $\cos x$ *when* $-0.5 \leq x \leq 0.5$. *Find an upper bound for the error. (This means: find a number* U *that you can show is bigger than the error.)*

Solution. The Alternating Series Estimation Theorem tells us that the magnitude of the error is at most that of the first unused term, which is $\frac{x^6}{6!}$. Since $|x| \leq 0.5$, the error is at most $\frac{0.5^6}{6!} \approx 0.0000217014$.

Ratio Test

1. What does the Ratio Test tell you about the following series?

   (a) $\displaystyle\sum_{k=0}^{\infty}(-1)^{k+1}\frac{1000^k}{k!}$.

   Solution. Since $\displaystyle\lim_{k\to\infty}\left|\frac{(-1)^{(k+1)+1}\frac{1000^{k+1}}{(k+1)!}}{(-1)^{k+1}\frac{1000^k}{k!}}\right| = \lim_{k\to\infty}\frac{1000^{k+1}\cdot k!}{(k+1)!\cdot 1000^k} = \lim_{k\to\infty}\frac{1000}{k+1} = 0$, the Ratio Test says that the series $\boxed{\text{converges absolutely}}$.

   (b) $\displaystyle\sum_{k=1}^{\infty}\frac{1}{k}$.

   Solution. $\displaystyle\lim_{k\to\infty}\left|\frac{\frac{1}{k+1}}{\frac{1}{k}}\right| = \lim_{k\to\infty}\left|\frac{k}{k+1}\right| = 1$, so the Ratio Test is $\boxed{\text{inconclusive}}$. (Of course, we know the series diverges.)

   (c) $\displaystyle\sum_{k=1}^{\infty}\frac{1}{k^2}$.

   Solution. $\displaystyle\lim_{k\to\infty}\left|\frac{\frac{1}{(k+1)^2}}{\frac{1}{k^2}}\right| = \lim_{k\to\infty}\left|\frac{k^2}{(k+1)^2}\right| = 1$, so the Ratio Test is $\boxed{\text{inconclusive}}$. (Of course, we know the series converges.)

2. *When we studied Taylor series, we found that the Taylor series for* $\sin x$ *about* 0 *was* $x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots$, *which can be written in summation notation as* $\displaystyle\sum_{k=0}^{\infty}(-1)^k\frac{x^{2k+1}}{(2k+1)!}$. *For what values of* x *does this series converge?*

   Solution. We'll use the Ratio Test:

$$\begin{aligned}\lim_{k\to\infty}\left|\frac{(-1)^{k+1}\frac{x^{2(k+1)+1}}{[2(k+1)+1]!}}{(-1)^k\frac{x^{2k+1}}{(2k+1)!}}\right| &= \lim_{k\to\infty}\left|\frac{x^{2k+3}\cdot(2k+1)!}{(2k+3)!\cdot x^{2k+1}}\right| \\ &= \lim_{k\to\infty}\left|\frac{x^2}{(2k+3)(2k+2)}\right| \\ &= 0.\end{aligned}$$

   (Since we're taking the limit as k tends to infinity, we treat x as a constant when taking the limit.) Therefore, the Ratio Test says that, no matter what x is, the series converges absolutely. That is, the series $\boxed{\text{converges absolutely for all } x}$.

3. *When we studied Taylor series, we found that the Taylor series for* $\ln(1+x)$ *about* 0 *was* $x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots$, *which can be written in summation notation as* $\displaystyle\sum_{k=1}^{\infty}(-1)^{k+1}\frac{x^k}{k}$. *For what values of* x *does this series converge?*

Solution. We use the Ratio Test:

$$\begin{aligned}
\lim_{k\to\infty}\left|\frac{(-1)^{(k+1)+1}\frac{x^{k+1}}{k+1}}{(-1)^{k+1}\frac{x^k}{k}}\right| &= \lim_{k\to\infty}\left|\frac{x^{k+1}\cdot k}{(k+1)\cdot x^k}\right| \\
&= \lim_{k\to\infty}\left|x\frac{k}{k+1}\right| \\
&= |x|
\end{aligned}$$

By the Ratio Test, the series converges absolutely when $|x| < 1$ and diverges when $|x| > 1$. However, the Ratio Test is inconclusive when $|x| = 1$, so we'll have to test $x = \pm 1$ separately.

When $x = 1$, the series is $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$. This is the alternating harmonic series, and we've seen that this converges conditionally. (See #1 from the "Absolute and Conditional Convergence" handout.)

When $x = -1$, the series is $-1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{4} - \cdots$, which is just -1 times the harmonic series, and we know that this diverges.

So, our final answer is:

$$\boxed{\sum_{k=1}^{\infty}(-1)^{k+1}\frac{x^k}{k}\quad\begin{cases}\text{converges absolutely} & \text{when } |x| < 1\\ \text{diverges} & \text{when } |x| > 1 \text{ or } x = -1\\ \text{converges conditionally} & \text{when } x = 1\end{cases}}$$

4. Decide whether the following series converge absolutely, converge conditionally, or diverge. You may use any method you like, but explain your reasoning. There is one that you will not be able to do (this is not due to a personal failing; it's just that all of the tests that we know are inconclusive).

   (a) $\displaystyle\sum_{n=1}^{\infty}\frac{\cos n}{n^2}$.

   Solution. When sin or cos appears in a series, it's often helpful to use the comparison test and the fact that $|\cos x|, |\sin x| \leq 1$. Remember that we need a series with positive terms to use the comparison test. So, let's look at the series of absolute values, $\displaystyle\sum_{n=1}^{\infty}\frac{|\cos n|}{n^2}$.

   Since $0 \leq \frac{|\cos n|}{n^2} \leq \frac{1}{n^2}$ and $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges (it's a p-series with $p = 2$), the Comparison Test says that $\displaystyle\sum_{n=1}^{\infty}\frac{|\cos n|}{n^2}$ converges.

   This tells us that $\displaystyle\sum_{n=1}^{\infty}\frac{\cos n}{n^2}$ $\boxed{\text{converges absolutely}}$.

   (b) $\displaystyle\sum_{n=100}^{\infty}\frac{n!n!}{(2n)!}$.

Solution. This has lots of factorials, so the Ratio Test is a good test to try.

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{\frac{(n+1)!(n+1)!}{(2n+2)!}}{\frac{n!n!}{(2n)!}}\right| &= \lim_{n\to\infty}\frac{(n+1)!(n+1)!(2n)!}{n!n!(2n+2)!} \\
&= \lim_{n\to\infty}\frac{(n+1)(n+1)}{(2n+1)(2n+2)} \\
&= \lim_{n\to\infty}\frac{n^2+2n+1}{4n^2+6n+2} \\
&= \frac{1}{4}
\end{aligned}$$

Therefore, the Ratio Test says that the series $\boxed{\text{converges absolutely}}$.

(c) $\displaystyle\sum_{n=0}^{\infty}\frac{\sin n}{n}$.

Solution. None of the tests we know work here.

(d) $\displaystyle\sum_{n=2}^{\infty}\frac{\ln n}{n}$.

Solution. We will use the Comparison Test: $0 \le \frac{1}{n} \le \frac{\ln n}{n}$ when $n > e$. We know that $\displaystyle\sum_{n=3}^{\infty}\frac{1}{n}$ diverges (it's the harmonic series), so the Comparison Test says that $\displaystyle\sum_{n=3}^{\infty}\frac{\ln n}{n}$ diverges, too. Adding on a term at the beginning doesn't affect convergence, so $\displaystyle\sum_{n=2}^{\infty}\frac{\ln n}{n}$ also $\boxed{\text{diverges}}$.

(e) $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n}{n^3+1}$.

Solution. When n is really big, $(-1)^{n+1}\frac{n}{n^3+1} \approx (-1)^{n+1}\frac{n}{n^3} = (-1)^{n+1}\frac{1}{n^2}$. We know that the series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges (it's a p-series with $p = 2$), so we can guess that the given series is probably absolutely convergent.

To verify this, we'll compare the series of absolute values $\displaystyle\sum_{n=1}^{\infty}\frac{n}{n^3+1}$ to $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}$ using the Limit Comparison Test (we are allowed to use the Limit Comparison Test because both of these series have positive terms; however, we may not use the Limit Comparison Test with the original series since it has both positive and negative terms).

Since $\displaystyle\lim_{n\to\infty}\frac{\frac{n}{n^3+1}}{\frac{1}{n^2}} = \lim_{n\to\infty}\frac{n^3}{n^3+1} = 1$ and $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges, $\displaystyle\sum_{n=1}^{\infty}\frac{n}{n^3+1}$ converges by the Limit Comparison Test. This means that $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n}{n^3+1}$ $\boxed{\text{converges absolutely}}$.

(f) $\displaystyle\sum_{n=5000}^{\infty}(-1)^n\frac{n}{n+1}$.

Solution. When n is really big, $(-1)^{n+1}\frac{n}{n+1} \approx (-1)^{n+1}\frac{n}{n} = (-1)^n$. These aren't going to 0, so we should use the Nth Term Test for Divergence: $\displaystyle\lim_{n\to\infty}(-1)^n\frac{n}{n+1}$ does not exist (half of the terms are getting closer to 1 while the other half are getting closer to -1), so the given series $\boxed{\text{diverges}}$.

Power Series

A <u>power series</u> centered at the number a is a series of the form $\sum_{n=0}^{\infty} c_n(x-a)^n$ where x is a variable and the c_n are constants.

1. For what values of x does the power series $\sum_{n=1}^{\infty} n!x^n$ converge? (This series is centered at 0.)

Theorem. For a given power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ centered at a, there are 3 possibilities:

1. The series converges only when $x = a$.
2. The series converges for all x.
3. There is a positive number R such that the series converges when $|x-a| < R$ and diverges when $|x-a| > R$. R is called the <u>radius of convergence</u>. (Note that this doesn't say anything about what happens when $|x-a| = R$.)

The <u>interval of convergence</u> of a power series is the set of x for which the power series converges.

2. Find the radius of convergence and interval of convergence of the power series $\sum_{n=1}^{\infty} \frac{2^n}{n}(x-3)^n$.

3. We know that the power series $\sum_{n=0}^{\infty} x^n$ converges to $\frac{1}{1-x}$ when $|x| < 1$. Find a power series representation of the function $\frac{x}{1+4x^2}$. What is the radius of convergence of this power series?

Theorem. If the power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ has radius of convergence R where $R > 0$ or $R = \infty$, then the function $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$ is differentiable on the interval $(a-R, a+R)$ and

1. $f'(x) = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1}$.

2. $\int f(x)\, dx = C + \sum_{n=0}^{\infty} \frac{c_n}{n+1}(x-a)^{n+1}$.

The power series in (1) and (2) both have radius of convergence R. (Note: Although the radius of convergence remains unchanged, the interval of convergence may change.)

4. (a) Find a power series representation for $\ln(1+x)$ centered at 0. What is the radius of convergence for the power series you have found? (Hint: What is the derivative of $\ln(1+x)$?)

   (b) Find the degree 5 Taylor polynomial approximation of $\ln(1+x)$.

5. Find a power series representation of $\arctan(5x)$ centered at 0. What is the radius of convergence of the power series you have found?

More on Power Series

1. Suppose we have a power series $\sum_{n=1}^{\infty} c_n (x+7)^n$.

   (a) *If you know that the power series converges when $x = 0$, what conclusions can you draw?*

   Solution. The power series is centered at -7, so the fact that it converges at $x = 0$ means that the interval of convergence is at least $(-14, 0]$.

   (b) *Suppose you also know that the power series diverges when $x = 1$. Now what conclusions can you draw?*

   Solution. The interval of convergence is at most $[-15, 1)$.

   (c) *Does $\sum_{n=1}^{\infty} c_n$ converge (assuming that the power series converges when $x = 0$ and diverges when $x = 1$)?*

   Solution. This is the power series when $x + 7 = 1$, or $x = -6$. In part (a), we found that -6 must be in the interval of convergence, so the series $\boxed{\text{converges}}$.

   (d) *Does $\sum_{n=1}^{\infty} c_n (-8.1)^n$ converge?*

   Solution. This is the power series when $x + 7 = -8.1$, or $x = -15.1$. In part (b), we found that -15.1 cannot be in the interval of convergence, so the series $\boxed{\text{diverges}}$.

   (e) *Does $\sum_{n=1}^{\infty} c_n (-8)^n$ converge?*

   Solution. This is the power series when $x+7 = -8$, or $x = -15$. Neither (a) nor (b) tells us what must happen, so there is $\boxed{\text{not enough information}}$ to determine whether the series converges.

2. (a) *Taking for granted that $\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$ for all x, find the Taylor series of $x \sin(x^3)$ at 0.*

   Solution. The theorem tells us that, if we can find a power series representation of $x \sin(x^3)$, then that is the Taylor series. So, rather than trying to find the Taylor series directly (by taking derivatives), let's look for a power series representation of $x \sin(x^3)$.

   We are told that

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \text{ for all } x.$$

   Replacing x by x^3 everywhere gives:

$$\sin x^3 = \sum_{n=0}^{\infty} (-1)^n \frac{x^{6n+3}}{(2n+1)!},$$

still true for all x since the first equation was true for all x. Now, we multiply both sides by x to get

$$x \sin x^3 = \boxed{\sum_{n=0}^{\infty} (-1)^n \frac{x^{6n+4}}{(2n+1)!}},$$

still for all x.

(b) *What is the radius of convergence of the power series you found in part (a)?*

Solution. We said in part (a) that the power series representation was valid for all x, so the radius of convergence of the series must be $\boxed{\infty}$.

(c) *Let $f(x) = x\sin(x^3)$. What is $f'''(0)$? $f^{(4)}(0)$?*

Solution. In part (a), we found that $f(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{6n+4}}{(2n+1)!}$. Unpacking the summation notation, $f(x) = x^4 - \frac{x^{10}}{3!} + \frac{x^{16}}{5!} - \cdots$. Using this fact, there are two ways to solve the problem.

The slick way: The theorem says that, since $f(x) = x^4 - \frac{x^{10}}{3!} + \frac{x^{16}}{5!} - \cdots$, $x^4 - \frac{x^{10}}{3!} + \frac{x^{16}}{5!} - \cdots$ is the Taylor series of $f(x)$ at 0. This means that the coefficient of the x^n term is $\frac{f^{(n)}(0)}{n!}$ (because this is the formula we used to find the coefficients of the Taylor series).

So, $\frac{f'''(0)}{3!} = 0$ (because 0 is the coefficient of the x^3 term in $x^4 - \frac{x^{10}}{3!} + \frac{x^{16}}{5!} - \cdots$) and $\frac{f^{(4)}(0)}{4!} = 1$ (because 1 is the coefficient of the x^4 term in $x^4 - \frac{x^{10}}{3!} + \frac{x^{16}}{5!} - \cdots$). Solving, we get $\boxed{f'''(0) = 0}$ and $\boxed{f^{(4)}(0) = 4!}$.

A slower, but equally valid method: Another way you can tackle this problem is to start again with $f(x) = x^4 - \frac{x^{10}}{3!} + \frac{x^{16}}{5!} - \cdots$ and just differentiate. We get:

$$\begin{aligned}
f(x) &= x^4 - \frac{x^{10}}{3!} + \frac{x^{16}}{5!} - \cdots \\
f'(x) &= 4x^3 - \frac{10x^9}{3!} + \frac{16x^{15}}{5!} - \cdots \\
f''(x) &= 4 \cdot 3x^2 - \frac{10 \cdot 9x^8}{3!} + \frac{16 \cdot 15x^{14}}{5!} - \cdots \\
f'''(x) &= 4 \cdot 3 \cdot 2x - \frac{10 \cdot 9 \cdot 8x^7}{3!} + \frac{16 \cdot 15 \cdot 14x^{13}}{5!} - \cdots \\
f^{(4)}(x) &= 4 \cdot 3 \cdot 2 - \frac{10 \cdot 9 \cdot 8 \cdot 7x^6}{3!} + \frac{16 \cdot 15 \cdot 14 \cdot 13x^{12}}{5!} - \cdots
\end{aligned}$$

If we plug in $x = 0$ to the last two equations, we get $f'''(0) = 0$ and $f^{(4)}(0) = 4 \cdot 3 \cdot 2$.

Note: Either of these two methods should be a lot faster than just starting with $f(x) = x\sin(x^3)$ and differentiating that 4 times; this is one advantage of being able to write $f(x)$ as a power series. Of course, if you wanted to know $f^{(100)}(0)$, the first method is going to be a lot faster than the second.

3. (a) *Find a power series representation of* $\arctan(5x)$ *centered at* 0.

Solution. Let's start by finding a power series representation of $\arctan x$. Then we can replace x by $5x$ to get a power series representation of $\arctan(5x)$.

We know that the derivative of $\arctan x$ is $\frac{1}{1+x^2}$.

We start with:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \text{ valid when } |x| < 1.$$

Let's replace x by $-x^2$:

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-x^2)^n, \text{ valid when } |-x^2| < 1.$$

We'll simplify the right side a little:

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n}, \text{ valid when } |-x^2| < 1.$$

Note that $|-x^2| = |x^2| = |x|^2$, so saying $|-x^2| < 1$ is the same as saying $|x|^2 < 1$, or $|x| < 1$. That is, the radius of convergence of the power series on the right is 1.

Now integrate both sides:

$$\arctan x = C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}.$$

Integrating a power series doesn't change the radius of convergence, so the radius of convergence of this power series is still 1.

We need to solve for the constant of integration C; we do this by plugging in $x = 0$ on both sides of the equation:

$$\arctan 0 = C + 0,$$

so $C = 0$. Therefore,

$$\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}.$$

Since the radius of convergence of this power series is 1, the power series converges when $|x| < 1$ and diverges when $|x| > 1$.

Finally, we replace x by $5x$:

$$\arctan 5x = \sum_{n=0}^{\infty} (-1)^n \frac{(5x)^{2n+1}}{2n+1}.$$

The power series converges when $|5x| < 1$ and diverges when $|5x| > 1$. It's nice to simplify this a little bit, so we end up with

$$\arctan 5x = \boxed{\sum_{n=0}^{\infty} (-1)^n \frac{5^{2n+1}}{2n+1} x^{2n+1}}.$$

(b) *What is the radius of convergence of the power series you found in part (a)?*

Solution. In part (a), we said that the power series converges when $|5x| < 1$ and diverges when $|5x| > 1$. In other words, the power series converges when $|x| < \frac{1}{5}$ and diverges when $|x| > \frac{1}{5}$, so the radius of convergence is $\boxed{\frac{1}{5}}$.

4. In each part, find a power series that has the given interval of convergence. (Hint: If you get stuck, try finding the interval of convergence of $\sum_{n=1}^{\infty} \frac{x^n}{n}$.)

(a) $(-6, 0)$.

Solution. We know the geometric series $\sum_{k=0}^{\infty} x^k$ converges when $|x| < 1$ and diverges when $|x| \geq 1$. We're looking for something that converges when $|x+3| < 3$ and diverges when $|x+3| \geq 3$. Another way of saying this is that we want something that converges when $\left|\frac{x+3}{3}\right| < 1$ and diverges when $\left|\frac{x+3}{3}\right| \geq 1$. The geometric series $\boxed{\sum_{k=0}^{\infty} \left(\frac{x+3}{3}\right)^k}$ works. (Of course, there are infinitely many other possible answers.)

(b) $(-1, 3)$.

Solution. Now we want something that converges when $\left|\frac{x+1}{2}\right| < 1$ and diverges when $\left|\frac{x+1}{2}\right| \geq 1$. One possibility is the geometric series $\boxed{\sum_{k=0}^{\infty} \left(\frac{x+1}{2}\right)^k}$.

(c) *Challenge:* $[-1, 3)$.

Solution. This is more difficult because we can't use a geometric series. (The interval of convergence of a geometric series never includes its endpoints, but here we want to include the left endpoint.) Rememeber that the times we've had series where one endpoint is included but the other is not is when one endpoint gives us the alternating harmonic series (convergent) and the other gives us the harmonic series (divergent). So, let's try to make a power series where plugging in $x = -1$ gives us the alternating harmonic series and plugging in $x = 3$ gives us the harmonic series.

Using the hint, we'll look at $\sum_{n=1}^{\infty} \frac{x^n}{n}$. This has an interval of convergence of $[-1, 1)$, so it's almost what we want. Let's try to stretch and translate the function so that its interval of converge will be $[-1, 3)$. First, we want to stretch it so that the radius of convergence is 2 instead of 1. To do this, we replace x by $\frac{x}{2}$: $\sum_{n=1}^{\infty} \frac{(x/2)^n}{n}$. Next, we want to shift the center to 1: to do this, we replace x by $x - 1$: $\sum_{n=1}^{\infty} \frac{((x-1)/2)^n}{n}$, or $\boxed{\sum_{n=1}^{\infty} \frac{(x-1)^n}{2^n \cdot n}}$.

Series Problems

1. Decide whether the following series converge or diverge. Explain your reasoning.

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{2n^5 + 500n^4 + n^3}{n^7 + 200n^6}.$

   (b) $\displaystyle\sum_{n=100}^{\infty} \frac{\sin n}{n^2}.$

2. Use a second degree Taylor polynomial to approximate $33^{1/5}$.

3. We define a function $f(x)$ by setting $f(x) = \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{\sqrt{n}2^n}$ for those x for which the series converges.

(a) Find the radius of convergence.

(b) Write a power series representation of $f'(x)$, the first derivative of f. Use it to find a series for $f'(1)$.

(c) Write out the first three non-zero terms of your series for $f'(1)$. At $x = 1$, is the function f increasing or decreasing? Explain.

In #1 - #8, write a differential equation that reflects the situation. Include an initial condition if the information is given.

1. *The population of a certain country increases at a rate proportional to the population size. Let $P = P(t)$ be the population at time t.*

   Solution. The rate of change is $\frac{dP}{dt}$, and we also know that the rate of change is proportional to P, so it's kP for some k. (We know k must be positive because the population is increasing.) So, $\boxed{\frac{dP}{dt} = kP}$.

2. *A snowball melts at a rate proportional to its surface area. At time 0, the snowball has a radius of 10 cm. Let $r = r(t)$ be the radius of the snowball at time t.*

   Solution. The first sentence can be written as $\boxed{\frac{dr}{dt} = k(4\pi r^2)}$, where k is a constant. (In this case, k is going to be negative because the rate of change should be negative.) The second sentence can be written as $\boxed{r(0) = 10}$.

3. *A yellow rubber duck is dropped out of the window of an apartment building at a height of 80 feet. Let $s = s(t)$ be the height of the duck above the ground at time t. (Gravity is the acceleration -32 ft/s^2.)*

   Solution. The rubber duck accelerates due to gravity, so $\boxed{s''(t) = -32}$. We know that it starts at 80 feet above the ground, so $\boxed{s(0) = 80}$. We also know that the duck is not moving at the beginning, so its initial velocity is 0. That is, $\boxed{s'(0) = 0}$.

4. *Ferdinand is trying to fill a bucket from a faucet. Unfortunately, he doesn't realize that there is a small hole in the bottom of the bucket. Water flows in to the bucket from the faucet at a constant rate of .75 quarts per minute, and it flows out of the hole at a rate proportional to the amount of water $W(t)$ already in the bucket (due to the increased water pressure).*

   Solution. The rate of change of water in the bucket is $\frac{dW}{dt}$, which is equal to (rate of water coming in) minus (rate of water coming out). The rate of water coming in is 0.75. The rate of water going out is $kW(t)$ where k is a positive constant. So, $\boxed{\frac{dW}{dt} = 0.75 - kW}$.

5. *A drug is being administered to a patient at a constant rate of c mg/hr. The patient metabolizes and eliminates the drug at a rate proportional to the amount in his body. Let $M = M(t)$ be the amount (in mg) of medicine in the patient's body at time t, where t is measured in hours.*

   Solution. The rate of change of medicine in the patient's body is equal to (rate in) minus (rate out). The rate in is c. The rate out is proportional to the amount in his body, so it's kM for some positive constant k. Therefore, the appropriate model is $\boxed{\frac{dM}{dt} = c - kM}$.

6. *\$6000 is deposited in a bank account. The account has a nominal annual interest rate of 2%, compounded continuously. There are no deposits and no withdrawals. Let $M = M(t)$ be the amount of money in the account at time t, where t is measured in years.*

Solution. If $t = 0$ is the time the money is deposited, then the first sentence is saying that $\boxed{M(0) = 6000}$. The second sentence is saying that $\boxed{\frac{dM}{dt} = .02M}$.

7. *\$6000 is deposited in a bank account. The account has a nominal annual interest rate of 2%, compounded continuously. Money is being withdrawn at a rate of \$500 per year.*[1] *Let $M = M(t)$ be the amount of money in the account at time t, where t is measured in years.*

   Solution. The rate at which money is leaving the account is 500. Everything else is the same as in the previous problem, so we have $\boxed{\frac{dM}{dt} = .02M - 500}$ and $\boxed{M(0) = 6000}$.

8. *A rumor spreads at a rate proportional to product of the number of people who have heard it and the number who have not. In a town of N people, suppose 1 person originates the rumor at time $t = 0$. Let $y = y(t)$ be the number of people who have heard the rumor at time t.*

   What does this model imply about the number of people who eventually have heard the rumor?

   Solution. The initial condition is $y(0) = 1$ since 1 person has heard the rumor at time 0. The rate of change is proportional to the product of the number of people who have heard it (y) and the number who have not ($N - y$), so $\boxed{\frac{dy}{dt} = ky(N - y)}$, where k is a positive constant.

   As long as not everybody has heard the rumor, the spreading rate is positive. Therefore, it seems like eventually everybody will have heard the rumor.

The following problems are about solutions to differential equations.

9. *Which of the following is a solution to $\frac{dy}{dx} = y$?*

   (a) $y = \frac{x^2}{2} + C$.

   (b) $y = e^x + C$.

   (c) $y = Ce^x$.

   Solution. $\boxed{\text{(c)}}$. If $y = Ce^x$, then $\frac{dy}{dx} = Ce^x = y$.

   If $y = \frac{x^2}{2} + C$, then $\frac{dy}{dx} = x$, which is not equal to y.

   If $y = e^x + C$, then $\frac{dy}{dx} = e^x$, which is not equal to y (unless $C = 0$).

10. *Give two solutions to $\frac{dy}{dx} = 5y$. What is the general solution?*

    Solution. A general solution is $\boxed{Ce^{5x}}$. Two specific solutions are e^{5x} and $-e^{5x}$.

11. *Give two solutions to to $\frac{dy}{dx} = 5x$. What is the general solution?*

    Solution. We actually know how to solve this already: the equation says we are looking for a function of x whose derivative is $5x$. That is, we want antiderivatives of $5x$, and we know that the general antiderivative is $\boxed{\frac{5}{2}x^2 + C}$. Two specific solutions are $\frac{5}{2}x^2$ and $\frac{5}{2}x^2 - 1$.

[1] In reality, you cannot withdraw money continuously from the bank, but it's convenient to use a continuous model.

Slope Fields

1. Draw the slope fields for the following differential equations:

   (a) $\frac{dy}{dt} = 1$.

   (b) $\frac{dy}{dt} = t$.

   (c) $\frac{dy}{dt} = y$.

   (d) $\frac{dy}{dt} = \frac{-t}{y}$.

2. Draw the slope field for the differential equation $\frac{dy}{dt} = y - 1$. Sketch two solutions to the equation.

3. Which of the following is a solution to $\frac{dy}{dt} = y - 1$?

   (a) $y = Ce^t$

   (b) $y = Ce^t - t$

   (c) $y = Ce^{-t} - 1$

   (d) $y = Ce^t - 1$

   (e) $y = Ce^t + 1$

4. Which of the following is a solution to $y'' - y' - 6y = 0$?

   (a) $y = Ce^t$.

   (b) $y = C \sin 2t$.

   (c) $y = 5e^{3t} + e^{-2t}$.

   (d) $y = e^{3t} - 2$.

1. *Find the general solution of the differential equation* $\frac{dM}{dt} = 2.4 - .2M$. *(Such a differential equation came up, for instance, when we modeled the amount of medicine in a patient's body.)*

   Solution. We can use separation of variables: $\frac{1}{2.4-0.2M}\ dM = dt$.[1] Simplifying,

$$-\frac{1}{0.2} \cdot \frac{1}{M-12}\ dM = dt.$$

   Integrating both sides,

$$-\frac{1}{0.2} \ln|M-12| = t + C.$$

   Multiplying both sides by -0.2 gives

$$\ln|M-12| = -0.2t - 0.2C.$$

   Since $-0.2C$ is just an arbitray constant, we can give it a new name; let's call it A. So,

$$\ln|M-12| = A - 0.02t.$$

   Then,

$$M - 12 = \pm e^A e^{-0.02t}.$$

   Again, $\pm e^A$ is just an arbitrary constant, so let's call it B. So,

$$M - 12 = Be^{-0.02t},$$

   and $\boxed{M(t) = 12 + Be^{-0.02t}}$.[2]

2. *Last time, we solved the differential equation* $\frac{dy}{dt} = -\frac{t}{y}$ *by drawing the slope field, guessing the solution, and checking it. Now, solve the differential equation using separation of variables.*

   Solution. We can rewrite $\frac{dy}{dt} = -\frac{t}{y}$ as

$$y\ dy = -t\ dt.$$

   Integrating both sides,

$$\frac{1}{2}y^2 = -\frac{1}{2}t^2 + C.$$

   Multiplying both sides by 2,

$$y^2 = -t^2 + 2C.$$

   Since $2C$ is still just an arbitrary constant, we can give it a new name; let's call it A. So,

$$y^2 = A - t^2,$$

   and $\boxed{y = \pm\sqrt{A - t^2}}$.

[1] Technically, we can only do this if $2.4 - 0.2M \neq 0$; if $2.4 - 0.2M = 0$, which happens when $M = 12$, the original differential equation is just $\frac{dM}{dt} = 0$, so $M(t) = 12$ is a solution.

[2] Since $B = \pm e^A$, B should technically be non-zero. But we remarked earlier that $M(t) = 12$ is a solution, so $B = 0$ is also okay.

3. *Solve the differential equation* $\frac{dy}{dt} = e^{-t-y}$, *and find the particular solution satisfying the initial condition* $y(0) = 1$.

Solution. We can rewrite $\frac{dy}{dt} = e^{-t-y}$ as

$$e^y \, dy = e^{-t} \, dt.$$

Integrating both sides,

$$e^y = -e^{-t} + C.$$

So, $y(t) = \ln(C - e^{-t})$. Plugging in the initial condition gives $1 = \ln(C-1)$, so $e = C - 1$, and $C = 1 + e$. So, our answer is $\boxed{y(t) = \ln(1 + e - e^{-t})}$.

4. *Solve the differential equation* $y' = 2y - 6$.

Solution. First, $y(t) = 3$ is a solution.

If $y \neq 3$, we can rewrite the differential equation as $\frac{dy}{dt} = 2y - 6$, or $\frac{1}{2y-6} \, dy = dt$. To integrate both sides, it's helpful to rewrite the left side as $\frac{1}{2} \cdot \frac{1}{y-3}$. So, we have

$$\begin{aligned}
\int \frac{1}{2}\frac{1}{y-3} \, dy &= \int \, dt \\
\frac{1}{2} \ln|y-3| &= t + C \\
\ln|y-3| &= 2t + A \text{ where } A = 2C \\
|y-3| &= e^A e^{2t} \\
y - 3 &= \pm e^A e^{2t} \\
y - 3 &= Be^{2t} \text{ where } B = \pm e^A \\
y &= \boxed{3 + Be^{2t}}
\end{aligned}$$

Here, B can be 0 (because we started out saying that $y(t) = 3$ is a solution), or it can be $\pm e^A$, which accounts for any positive or negative constant. So, B can be any constant.

5. *Which of the following differential equations are separable? (You need not solve.)*

   (a) $\frac{dy}{dt} = t + y$.

   (b) $\frac{dy}{dt} = \frac{y}{\sin t}$.

   (c) $\frac{dy}{dt} = \frac{\sin t}{y} + t$.

Solution. (a) is not separable.

(b) is separable, for we can rewrite it as $\frac{1}{y} \, dy = \frac{1}{\sin t} \, dt$.

(c) is not separable.

6. *A* 20*-quart juice dispenser in a cafeteria is filled with a juice mixture that is* 10% *mango and* 90% *orange juice. A pineapple-mango blend (*40% *pineapple and* 60% *mango) is entering the dispenser at a rate of* 4 *quarts an hour and the well-stirred mixture leaves at a rate of* 4 *quarts an hour. Model the situation with a differential equation whose solution,* $M(t)$*, is the amount of mango juice in the container at time* t*. (*$t = 0$ *is the time when the pineapple-mango blend starts to enter the dispenser.)*

Solution. Since $M(t)$ is the amount of mango juice in the container at time t, $\frac{dM}{dt}$ is the rate of change of the amount of mango juice in the container. We know that this is equal to (rate at which mango juice is entering the container) minus (rate at which mango juice is leaving the container).

Let's first focus on the stuff entering the container. This is a pineapple-mango blend, entering at a rate of 4 quarts per hour. Only 60% of this is mango juice though, so mango juice is entering the container at a rate of $.6 \cdot 4 = 2.4$ quarts per hour.

Now, let's look at the stuff exiting the container. This is a blend of all of the juices, and it's leaving at a rate of 4 quarts per hour. What we need to know is what percent of this blend is mango juice. This is simple if we think about what our variables mean: $M(t)$ is the amount of mango juice in the container at time t, while there is always 20 quarts of juice in all. So, the percentage of the 20 quarts which is mango juice is $\frac{M(t)}{20}$. Therefore, mango juice is leaving the container at a rate of $\frac{M(t)}{20} \cdot 4 = \frac{M(t)}{5}$ quarts per hour.

So, our final differential equation is $\boxed{\frac{dM}{dt} = 2.4 - \frac{M}{5}}$. We also have an initial condition, since we know how much mango juice is in the dispenser at the beginning: 10% of the 20 quarts, or 2 quarts. So, our initial condition is $M(0) = 2$.

7. *Suppose that, in the previous problem, the mixture was leaving at a rate of 5 quarts per hour rather than 4 quarts per hour. Model the new situation.*

Solution. We still need to use $\frac{dM}{dt}$ = (rate at which mango juice is entering the container) minus (rate at which mango juice is leaving the container), and the rate at which mango juice is entering the container is just like in the previous problem, 2.4 quarts per hour.

Let's look at the stuff exiting the container. Again, we need to use the formula (rate at which mango juice is leaving) = (concentration of mango juice in the mixture) times (rate at which mixture is leaving). The rate at which the mixture is leaving is 5 quarts per hour. The concentration of mango juice in the mixture is equal to (amount of mango juice in mixture) divided by (total amount of mixture). The amount of mango juice is exactly $M(t)$. The total amount of mixture is a little more complicated. At time $t = 0$, there is 20 quarts of juice in the container. However, because the juice is entering at a rate of 4 quarts per minute and leaving at 5 quarts per minute, there is a net loss of 1 quart per minute. Thus, after t minutes, the amount of juice in the container is $20 - t$. So, the concentration of mango juice in the mixture at time t is $\frac{M(t)}{20-t}$, and the rate at which mango juice is leaving the container is $\frac{M(t)}{20-t} \cdot 5$.

So, our final differential equation is $\boxed{\frac{dM}{dt} = 2.4 - \frac{5M}{20-t}}$. Again, we have the initial condition $M(0) = 2$.

Second-Order Homogeneous Differential Equations with Constant Coefficients

1. Suppose $f(t)$ and $g(t)$ are both solutions to the differential equation $y''+by'+cy=0$. Is $C_1f(t)+C_2g(t)$ a solution as well?

2. Can you guess solutions of $y''=y$? Try to guess two solutions that are not just multiples of each other.

3. Can you guess solutions of $y''=4y$? Try to guess two solutions that are not just multiples of each other.

4. Solve $y''-y'=6y$.

5. Solve $y'' + 5y' + 4y = 0$ where $y(0) = 1$ and $y'(0) = 2$.

6. Solve $y'' - 4y' + 4y = 0$.

7. Show that, if the characteristic equation $y'' + by' + cy = 0$ has one repeated root r, then $y = te^{rt}$ is a solution to $y'' + by' + cy = 0$.

1. *Solve* $y'' - 6y' + 9y = 0$.

   Solution. The characteristic equation is $r^2 - 6r + 9 = 0$, or $(r-3)^2 = 0$. Since $r = 3$ is a repeated root of this equation, the general solution is $\boxed{C_1 e^{3t} + C_2 t e^{3t}}$.

2. *Solve* $y'' + y = 0$.

   Solution. The characteristic equation is $r^2 + 1 = 0$, and the roots are $r = \pm i$. Since $e^{it} = \cos t + i \sin t$ and $e^{-it} = \cos(-t) + i\sin(-t) = \cos t - i \sin t$, two different solutions are $\cos t$ and $\sin t$. Thus, the general solution is $\boxed{y(t) = C_1 \cos t + C_2 \sin t}$.

3. *Solve* $y'' - 2y' + 5y = 0$.

   Solution. The characteristic equation is $r^2 - 2r + 5 = 0$. The roots of this are

$$\begin{aligned} \frac{2 \pm \sqrt{2^2 - 4(1)(5)}}{2} &= \frac{2 \pm \sqrt{-16}}{2} \\ &= \frac{2 \pm 4i}{2} \\ &= 1 \pm 2i \end{aligned}$$

   So, we know two solutions are $e^{(1+2i)t}$ and $e^{(1-2i)t}$, and the general solution is $C_1 e^{(1+2i)t} + C_2 e^{(1-2i)t}$. We're only interested in the real solutions, so let's rewrite $e^{(1+2i)t}$ and $e^{(1-2i)t}$ to find the real solutions:

$$\begin{aligned} e^{(1+2i)t} &= e^{t+2it} \\ &= e^t \cdot e^{i(2t)} \\ &= e^t(\cos 2t + i \sin 2t) \\ &= e^t \cos 2t + i e^t \sin 2t \end{aligned}$$

   Similarly, $e^{(1-2i)t} = e^t \cos 2t - i e^t \sin 2t$.

   Our general solution was $C_1 e^{(1+2i)t} + C_2 e^{(1-2i)t}$, and we now know that we can rewrite this as $C_1 \left(e^t \cos 2t + i e^t \sin 2t\right) + C_2 \left(e^t \cos 2t - i e^t \sin 2t\right)$. Re-grouping the terms, we see that we can write this as $\boxed{A_1 e^t \cos 2t + A_2 e^t \sin 2t}$.

4. (a) *Solve* $y'' + 2y' + 4y = 0$ *with initial conditions* $y(0) = 1$ *and* $y'(0) = 0$.

   Solution. The characteristic equation $r^2 + 2r + 4 = 0$ has roots $r = \frac{-2 \pm \sqrt{4-16}}{2} = -1 \pm \sqrt{3}i$. Therefore, two different solutions are $e^{(-1+\sqrt{3}i)t}$ and $e^{(-1-\sqrt{3}i)t}$. We rewrite:

$$\begin{aligned} e^{(-1+\sqrt{3}i)t} &= e^{-t+\sqrt{3}ti} \\ &= e^{-t} \cdot e^{i(\sqrt{3}t)} \\ &= e^{-t}\left[\cos\left(\sqrt{3}t\right) + i \sin\left(\sqrt{3}t\right)\right] \end{aligned}$$

   Following the same reasoning as in the previous problem, we see that the general solution is $y(t) = C_1 e^{-t} \cos\left(\sqrt{3}t\right) + C_2 e^{-t} \sin\left(\sqrt{3}t\right)$.

   Let's use the initial conditions to solve for C_1 and C_2. The initial condition $y(0) = 1$ tells us $C_1 \cos(0) + C_2 \sin(0) = 1$. Since $\cos(0) = 1$ and $\sin(0) = 0$, we know that $C_1 = 1$.

To use the second initial condition, we first need to differentiate $y(t)$:

$$y'(t) = C_1\left[-e^{-t}\cos\left(\sqrt{3}t\right) - \sqrt{3}e^{-t}\sin\left(\sqrt{3}t\right)\right] + C_2\left[-e^{-t}\sin\left(\sqrt{3}t\right) + \sqrt{3}e^{-t}\cos\left(\sqrt{3}t\right)\right].$$

Therefore, $y'(0) = -C_1 + \sqrt{3}C_2$. So, $0 = -1 + \sqrt{3}C_2$, and $C_2 = \frac{1}{\sqrt{3}}$.

Thus, our final solution is $\boxed{y(t) = e^{-t}\cos\left(\sqrt{3}t\right) + \frac{1}{\sqrt{3}}e^{-t}\sin\left(\sqrt{3}t\right)}$.

(b) *Interpret part (a) in terms of a vibrating spring. What is happening to the spring as time goes on?*

Solution. The differential equation $y'' + 2y' + 4y = 0$ describes a vibrating spring with friction. The initial condition $y(0) = 1$ says that the spring is initially stretched 1 unit beyond its equilibrium position, and the initial condition $y'(0) = 0$ says that its initial velocity is 0.

Our solution was $y(t) = e^{-t}\cos\left(\sqrt{3}t\right) + \frac{1}{\sqrt{3}}e^{-t}\sin\left(\sqrt{3}t\right)$, or $y(t) = e^{-t}\left[\cos\left(\sqrt{3}t\right) + \frac{1}{\sqrt{3}}\sin\left(\sqrt{3}t\right)\right]$. This function oscillates while decreasing in magnitude, and $\lim_{t\to\infty} y(t) = 0$.

5. *Which of the following differential equations has periodic solutions? What is the period?*

(a) $y'' + 2y' - 3y = 0$.

(b) $y'' + 2y + 3y = 0$.

(c) $y'' + 4y' = 0$.

(d) $y'' + 4y = 0$.

(e) $y'' - 4y = 0$.

Does this agree with your interpretation of the differential equations in terms of vibrating springs?

Solution. Let's look at each one.

(a) The characteristic equation is $r^2 + 2r - 3 = 0$, or $(r+3)(r-1) = 0$. So, the general solution is $C_1e^{-3t} + C_2e^t$, which is not periodic. This differential equation can't be interpreted in terms of vibrating springs ($y'' + by' + cy = 0$ is only the differential equation for a vibrating spring if $b \geq 0$ and $c > 0$).

(b) The characteristic equation is $r^2 + 2r + 3 = 0$, which has roots $r = \frac{-2\pm\sqrt{-8}}{2} = -1 \pm \sqrt{2}i$. Since $e^{(-1+\sqrt{2}i)t} = e^{-t} \cdot e^{i(\sqrt{2}t)} = e^{-t}(\cos\sqrt{2}t + i\sin\sqrt{2}t)$, if we follow the reasoning we used in #3, we find that the general solution is $C_1e^{-t}\cos(\sqrt{2}t) + C_2e^{-t}\sin(\sqrt{2}t)$. This is not periodic. This differential equation describes a vibrating spring subject to friction (the fact that $b > 0$ means there is friction), so it makes sense that the solution is not periodic. (Such a spring should vibrate less and less over time rather than vibrating the same amount forever.)

(c) The characteristic equation is $r^2 + 4r = 0$, or $r(r+4) = 0$. This has roots $r = 0$ and $r = -4$, so the general solution is $C_1 + C_2e^{-4t}$, which is not periodic. This differential equation can't be interpreted in terms of vibrating springs.

(d) The characteristic equation is $r^2 + 4 = 0$, or $r = \pm 2i$. Since $e^{2it} = \cos 2t + i\sin 2t$, if we follow the reasoning we used in #3, we find that the general solution is $C_1\cos(2t) + C_2\sin(2t)$. This is periodic with period π. This differential equation describes a vibrating spring without friction (since $b = 0$), and it makes sense that such a spring should oscillate back and forth periodically.

(e) The characteristic equation is $r^2 - 4 = 0$, or $(r-2)(r+2) = 0$. This has roots $r = 2$ and $r = -2$, so the general solution is $C_1 e^{-2t} + C_2 e^{-2t}$, which is not periodic. This differential equation can't be interpreted in terms of vibrating springs.

Second-Order Homogeneous Differential Equations with Constant Coefficients

1. Which of the following differential equations has periodic solutions? What is the period?

   (a) $y'' + 2y' - 3y = 0$.

   (b) $y'' + 2y' + 3y = 0$.

   (c) $y'' + 4y' = 0$.

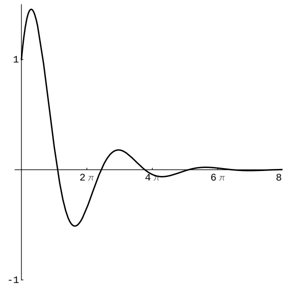
   (d) $y'' + 4y = 0$.

   (e) $y'' - 4y = 0$.

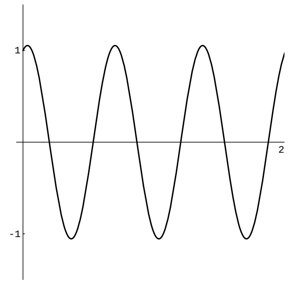
   Does this agree with your interpretation of the differential equations in terms of vibrating springs?

2. A spring with a mass of 5 kg has a natural length of 6 cm. A 20 N force is required to compress it to a length of 5 cm. If the spring is stretched to a length of 7 cm and released, find the position of the mass at time t. Sketch a graph of the position vs. time.

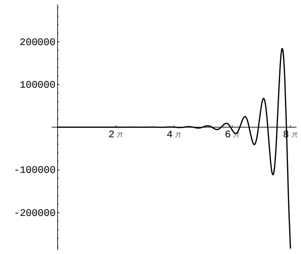
3. Match each differential equation with the graph of its solution. In each case, the differential equation has initial conditions $y(0) = 1$, $y'(0) = 1$.



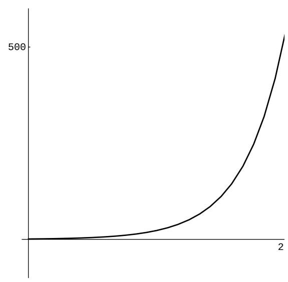
(1)



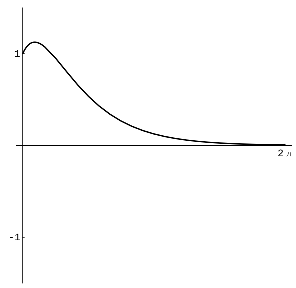
(2)



(3)



(4)



(5)

(a) $y'' + 3y' + 2y = 0.$

(b) $y'' + 9y = 0.$

(c) $y'' - 2y' + y = 0.$

(d) $y'' - y' + 10y = 0.$

(e) $y'' + \frac{1}{2}y' + \frac{5}{8}y = 0.$

4. Solve the differential equation $y'' + y = 0$ with initial conditions $y(0) = 0$ and $y'(0) = 1$.

Systems of Differential Equations

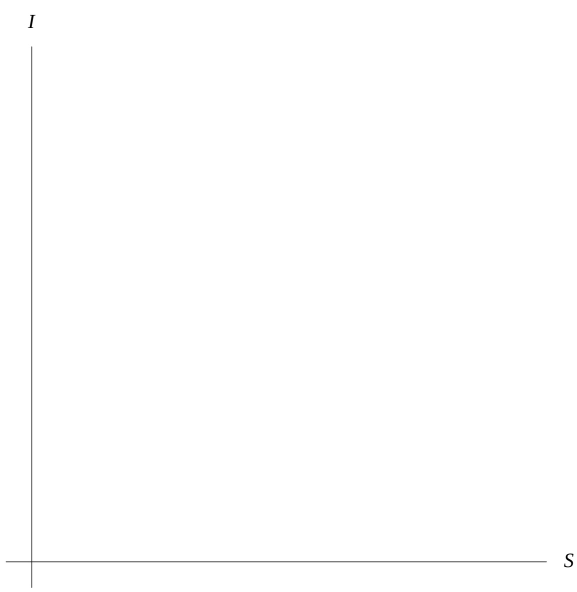
We've used systems of differential equations to model interaction between species. Systems can also be used to model disease epidemics.

Suppose that there is a large population of people, and some of the people have a fatal disease. This disease is infectious, so anybody who doesn't have the disease is susceptible to getting it. Let $I(t)$ be the number of people infected at time t, and let $S(t)$ be the number of people who are susceptible at time t.

1. How could you model this situation with a system of differential equations? You may ignore birth and death, except for death due to the disease, which you should include. (There are many many different answers; when in doubt, opt for simplicity.)

2. Using common sense, find the equilibrium points in this model. (You do not need to use the differential equations you found in #1; just think about the situation.)

3. Using common sense, sketch some typical phase trajectories in the phase plane.



4. A reasonable system for the situation described is:

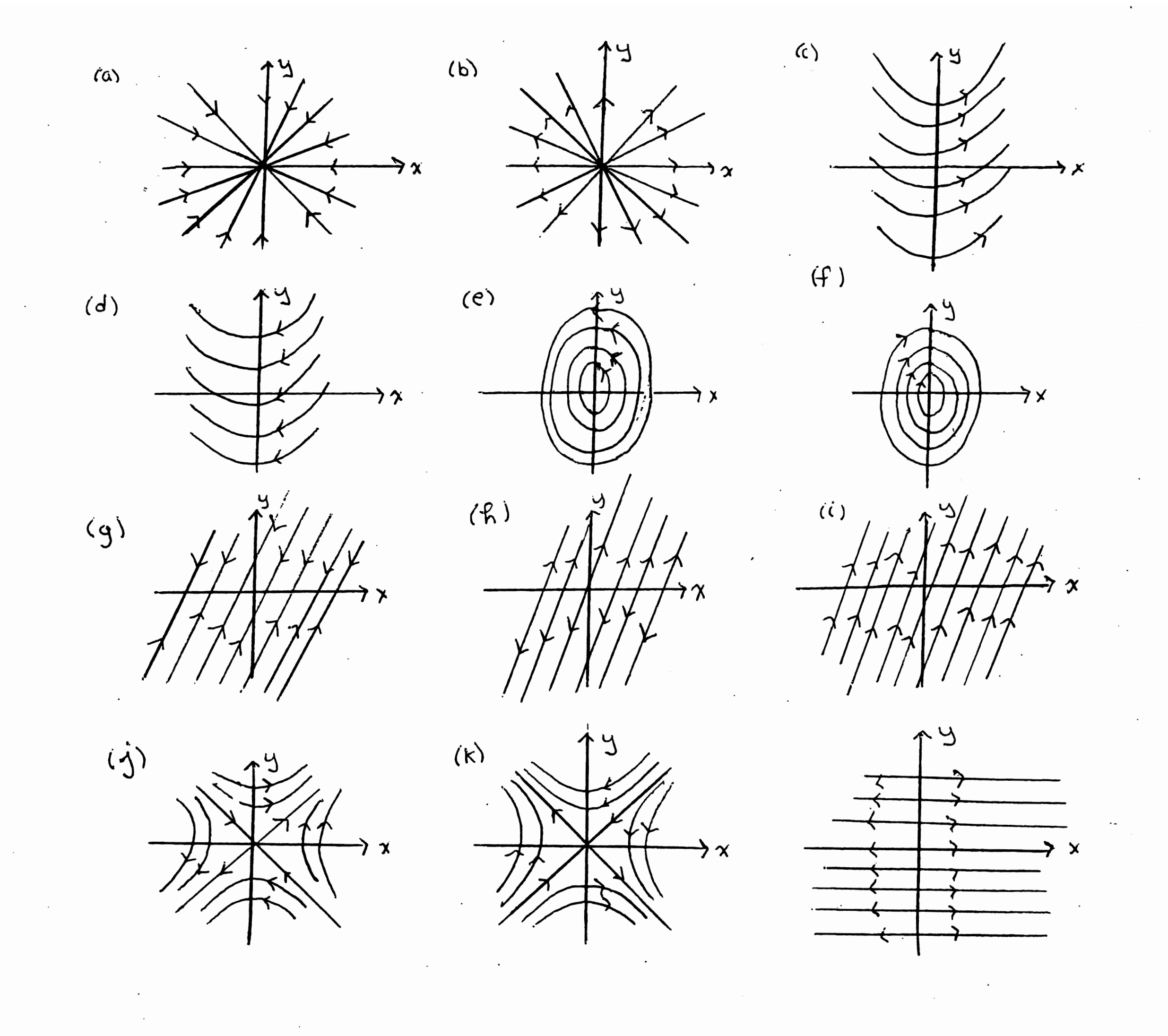
$$\begin{aligned}\frac{dS}{dt} &= -0.001IS \\ \frac{dI}{dt} &= 0.001IS - 0.1I\end{aligned}$$

Sketch the phase portrait for this system. (Be sure to draw the nullclines and equilibrium points.)

5. If the population starts with 50 infected people and 200 susceptible people, what will happen in the long run?

Systems of Differential Equations

The problems refer to these diagrams:

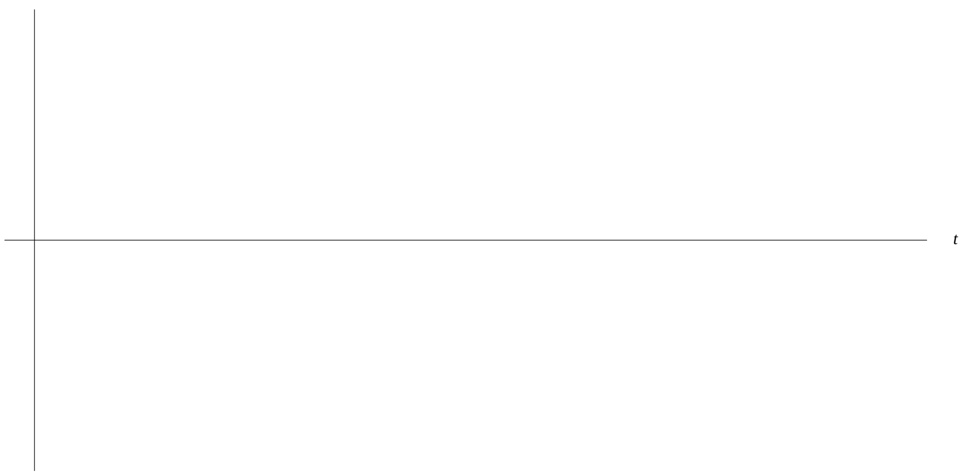


1. Find the diagram which matches the system.

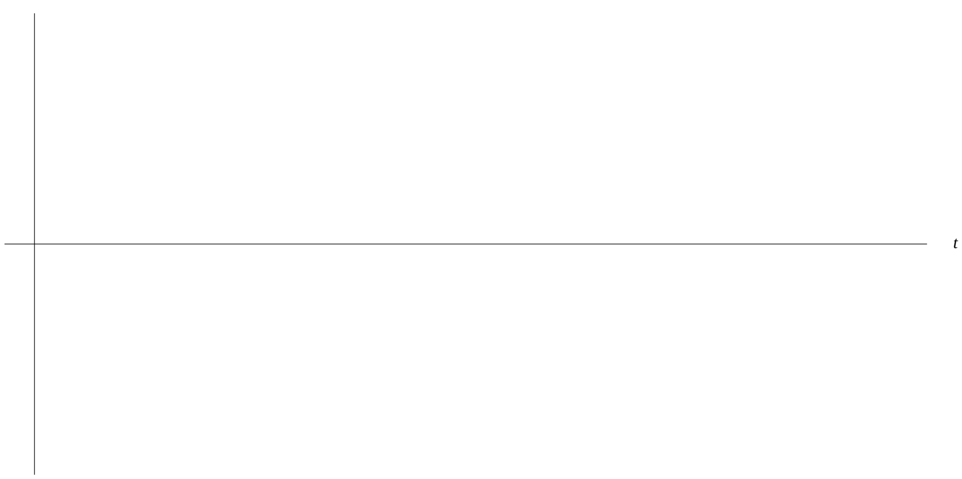
   (i) $\frac{dx}{dt} = y$ and $\frac{dy}{dt} = -2x$.

   (ii) $\frac{dx}{dt} = y$ and $\frac{dy}{dt} = 3y$.

2. (i) Suppose that the system $\frac{dx}{dt} = f(x, y)$ and $\frac{dy}{dt} = g(x, y)$ has phase portrait (e). Sketch possible graphs of $x(t)$ and $y(t)$, assuming $x(0) = 0$ and $y(0) = 3$.



   (ii) Do the same thing for (g).



3. Solve the system $\frac{dx}{dt} = y$ and $\frac{dy}{dt} = 3y$ with initial conditions $x(0) = 0$ and $y(0) = 3$.

4. Solve the system $\frac{dx}{dt} = y$ and $\frac{dy}{dt} = -2x$ with initial conditions $x(0) = 0$ and $y(0) = 3$.

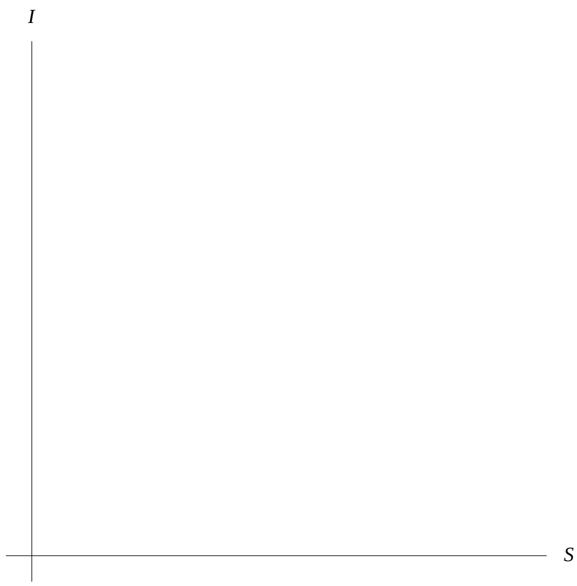
5. Find a system of differential equations whose phase portrait looks like (l), the last diagram.

Measles

We'll look at the system

$$\frac{dS}{dt} = -IS + 50$$
$$\frac{dI}{dt} = IS - 10I$$

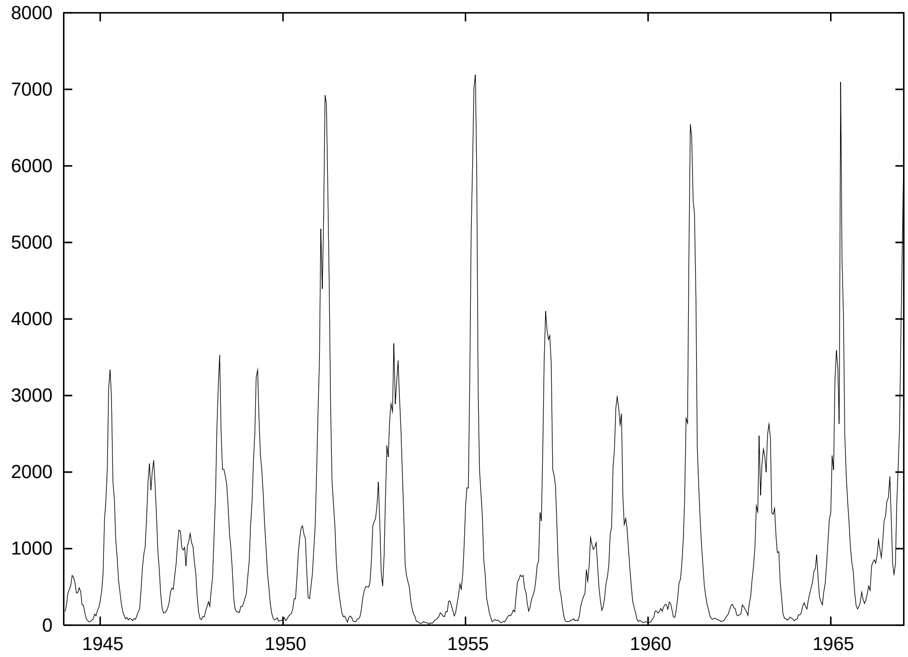
1. Do a qualitative phase plane analysis of this system. (You should draw equilibrium points, nullclines, and the direction of the trajectories in each region.)



2. Based on your phase plane analysis, what do you think the trajectories look like? Sketch a possible trajectory on your diagram if $S(0) = 5$ and $I(0) = 20$.

3. Using your trajectory, sketch a possible graph of $I(t)$ if $S(0) = 5$ and $I(0) = 20$.

4. This graph shows the number of cases of measles in 2 week periods in London from 1944 to 1966. Does our system give the same qualitative behavior?

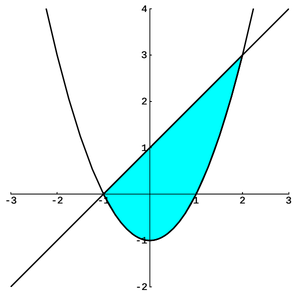


1. *The definite integral is defined to be*

   (a) *a limit of Riemann sums.*
   (b) *the difference in the evaluation of an antiderivative at the endpoints of the interval.*
   (c) *a signed area.*
   (d) *all of the above.*

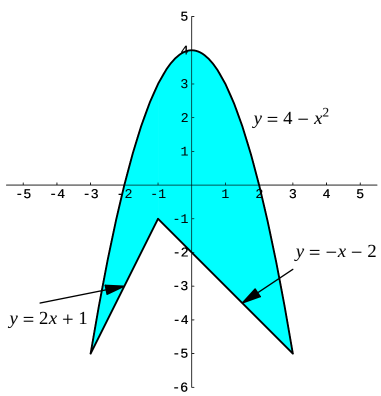
   Solution. $\boxed{\text{(a)}}$. We often *interpret* the definite integral as signed area, and we *compute* the definite integral using (b). However, neither of those is the *definition* of the definite integral.

2. *Write an integral or a sum and/or difference of integrals that gives the area enclosed by the graphs of* $y = x^2 - 1$ *and* $y = x + 1$.

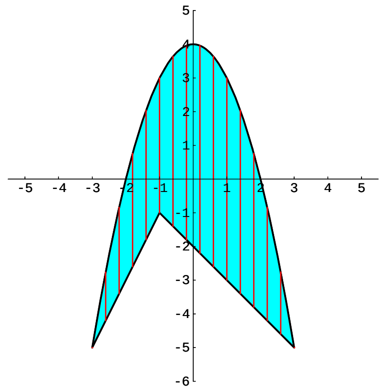


   Solution. The simplest answer is $\boxed{\int_{-1}^{2} [(x+1) - (x^2 - 1)]\ dx}$.

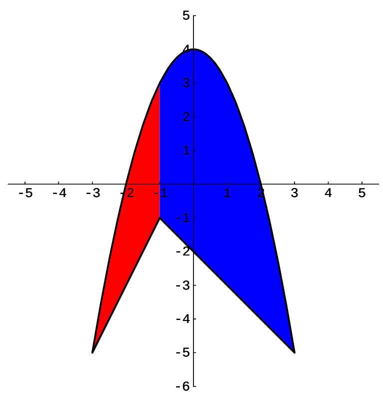
3. *Find the area of the following region.*



   Solution. According to our general theme, we should try to slice, approximate, sum, and take a limit. Let's try slicing vertically:

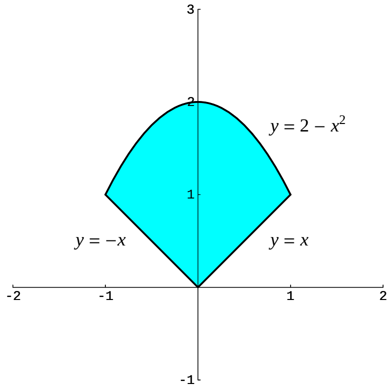


Notice that there are two "types" of slices in this picture. When x is in the interval $[-3,-1]$, the slices have height $(4-x^2)-(2x+1)$. When x is in the interval $[-1,3]$, the slices have height $(4-x^2)-(-x-2)$. We should really think about these two cases separately:



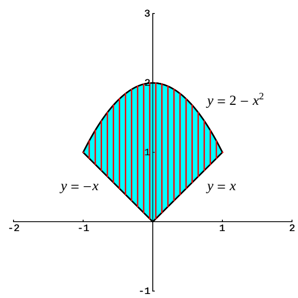
We'll find the area of the red piece and the blue piece separately and then just add those together. As we've already said, in the red piece, the slices have height $(4-x^2)-(2x+1)$, so the area of the red piece is $\int_{-3}^{-1}[(4-x^2)-(2x+1)]\;dx$. In the blue piece, the slices have height $(4-x^2)-(-x-2)$, so the area of the blue piece is $\int_{-1}^{3}[(4-x^2)-(-x-2)]\;dx$. Thus, the total area is $\boxed{\int_{-3}^{-1}[(4-x^2)-(2x+1)]\;dx+\int_{-1}^{3}[(4-x^2)-(-x-2)]\;dx}$. We can use the Fundamental Theorem of Calculus to evaluate these integrals, and we get $\boxed{\frac{38}{3}}$.

1. *Find the area of the region shown. (You may leave your answer as an integral or sum/difference of integrals.)*

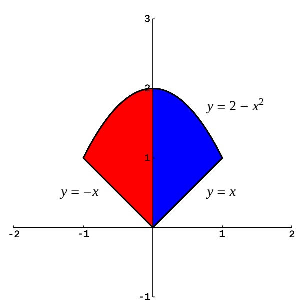


Solution. Remember that the basic idea in every integration problem is: slice, approximate, sum, and take a limit. What this boils down to is finding an expression for what's going on with the k-th slice. In this case, we're looking for area, so we want to slice the region and then approximate the area of the k-th slice.

There is a slight hitch though. Let's imagine that we slice the region vertically:

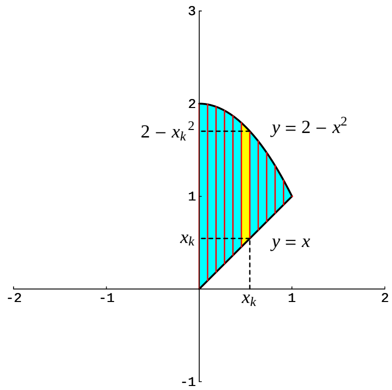


The problem is that we can't write a good description for the k-th slice. If the slice is on the left side of the y-axis, then its height would be about $(2 - x_k^2) - (-x_k)$. If the slice is on the right side, then its height is about $(2 - x_k^2) - x_k$. The solution to this problem is to break the region into two pieces and find their areas separately.



Actually, since the picture is symmetric, we can see right away that the red area is equal to the blue area. So, we can get by just finding the blue area and multiplying that by 2 to get the final answer.

So, let's focus on the right side. As usual, we slice into n pieces of equal width Δx.



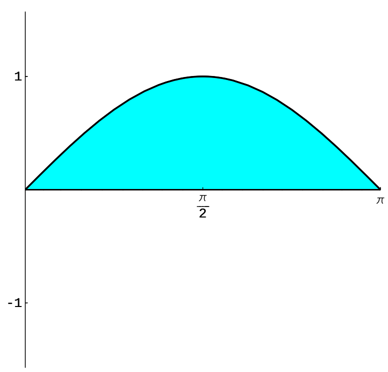
We can pretend that the k-th slice is a rectangle of width Δx and height $(2 - x_k^2) - x_k$, so we end up with

$$\boxed{\text{area of the } k\text{-th slice} \approx [(2 - x_k^2) - x_k]\Delta x}$$

Then we know that the limit of Riemann sums is $\lim_{n\to\infty} \sum_{k=1}^{n} [(2 - x_k^2) - x_k]\Delta x$, which is equal to the integral $\int_0^1 [(2 - x^2) - x]\, dx$.

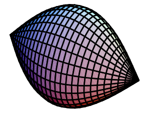
Finally, remember that this expression only represents half of the original area, so our final answer is $\boxed{2\int_0^1 [(2 - x^2) - x]\, dx}$.

2. Here is one loop of the sine curve.

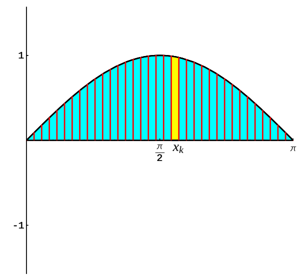


(a) *If you rotate this region about the x-axis, what shape do you get? What is its volume? (You do not need to evaluate your integral.)*

Solution. The solid looks like a football.



To find its volume, we slice and figure out what's happening with the k-th slice.



The k-th slice is very close to being a disk with radius $\sin x_k$ and thickness Δx, so

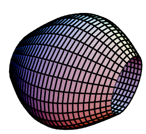
$$\boxed{\text{volume of } k\text{-th slice} \approx \pi(\sin x_k)^2 \Delta x}$$

Adding these up and taking the limit gives $\lim_{n\to\infty} \sum_{k=1}^{n} \pi(\sin x_k)^2 \Delta x$, which is equal to the integral

$\boxed{\int_0^\pi \pi(\sin x)^2 \, dx}$.

(b) *If you rotate the region about the line $y = -1$, what shape do you get? What is its volume? (You do not need to evaluate your integral.)*

Solution. This looks like a bead.



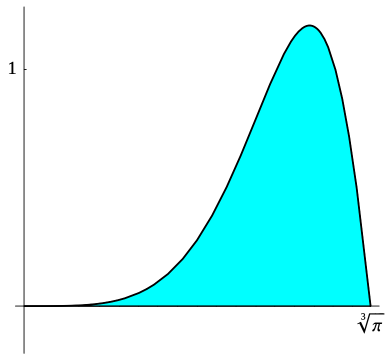
Again, to find its volume, we slice and figure out what's happening with the k-th slice. Now, the k-th slice isn't a disk any more; it's close to being a washer (disk with a round hole cut out). The outer radius of the washer is $\sin x_k + 1$, the inner radius is 1, and the thickness is Δx. Therefore,

$$\boxed{\text{volume of } k\text{-th slice} \approx \pi[(\sin x_k + 1)^2 - 1^2]\Delta x}$$

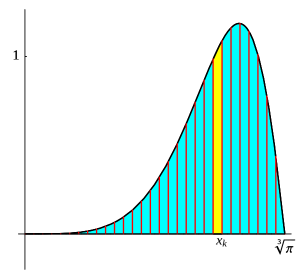
Adding these up and taking the limit gives us $\lim_{n\to\infty} \sum_{k=1}^{n} \pi[(\sin x_k + 1)^2 - 1^2]\Delta x$, which is equal to the integral $\boxed{\int_0^\pi \pi[(\sin x + 1)^2 - 1] \, dx}$.

More on Volumes

1. *This is the curve $y = x \sin x^3$. If we rotate this region about the y-axis, what is the volume of the resulting solid? (Once you get an integral, try to evaluate it.)*



Solution. We may either slice the region horizontally or vertically. Slicing horizontally will give us washers, but to find the inner radius, we would need to solve $y = x \sin x^3$ for x. We don't really know how to do that, so we had better use vertical slices instead.



When we rotate the k-th slice about the y-axis, we get a hollow tube (like a paper towel tube). The radius of this tube is x_k, the height is $x_k \sin x_k^3$, and the thickness is Δx. Therefore, the volume of the tube is approximately $2\pi x_k(x_k \sin x_k^3)\Delta x = 2\pi x_k^2 \sin x_k^3 \Delta x$.

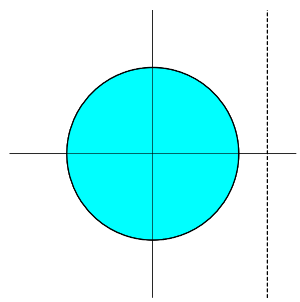
To approximate the volume of the whole solid, we have to sum up the volumes of all of the slices (slice 1 through slice n), which gives $\sum_{k=1}^{n} 2\pi x_k^2 \sin x_k^3 \Delta x$. To make the approximation accurate, we need to use more and more slices, so we take the limit as $n \to \infty$. So, our answer is $\lim_{n\to\infty} \sum_{k=1}^{n} 2\pi x_k^2 \sin x_k^3 \Delta x$, which is the integral $\int_0^{\sqrt[3]{\pi}} 2\pi x^2 \sin x^3 \, dx$. We can evaluate this integral using substitution. Let $u = x^3$.

Then, $du = 3x^2\ dx$. Since x goes from 0 to $\sqrt[3]{\pi}$, u goes from 0 to π. So,

$$\begin{aligned}\int_0^{\sqrt[3]{\pi}} 2\pi x^2 \sin x^3\ dx &= \int_0^{\pi} 2\pi \sin u \frac{du}{3} \\ &= \frac{2\pi}{3}\int_0^{\pi} \sin u\ du \\ &= \frac{2\pi}{3}(-\cos u)\Big|_0^{\pi} \\ &= \frac{2\pi}{3}(-\cos\pi + \cos 0) \\ &= \boxed{\frac{4\pi}{3}}\end{aligned}$$

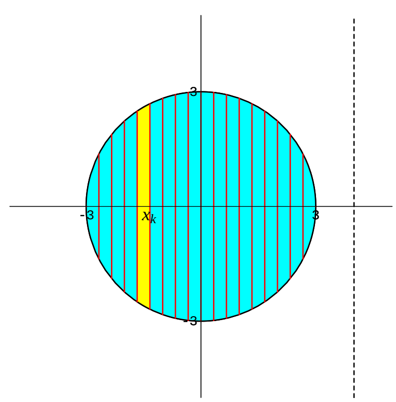
2. *How can you describe a bagel as a solid of revolution? (That is, what sort of region would you rotate, and what line would you rotate it about?)*

Solution. We can get a bagel by rotating a disk around a line, something like this (rotate the disk around the dotted line):



3. *The disk of radius 3 centered at the origin is rotated about the line $x = 4$. Find the volume using vertical slices. (It is also possible to do it using horizontal slices, and you might want to try that for extra practice.)*

Solution. Here are the slices.



The equation of the circle is $x^2 + y^2 = 9$, or $y = \pm\sqrt{9-x^2}$. $y = \sqrt{9-x^2}$ is the equation of the top half of the circle, and $y = -\sqrt{9-x^2}$ is the equation of the bottom half of the circle.

Rotating the k-th slice gives (approximately) a paper towel tube with thickness Δx, radius $4 - x_k$, and height $2\sqrt{9-x_k^2}$. So, the volume of the k-th piece is approximately $2\pi(4-x_k)(2\sqrt{9-x_k^2})\Delta x = 4\pi(4-x_k)\sqrt{9-x_k^2}\Delta x$.

Adding these up and taking the limit, we get the integral $\int_{-3}^{3} 4\pi(4-x)\sqrt{9-x^2}\,dx = 4\pi\int_{-3}^{3}(4-x)\sqrt{9-x^2}\,dx$. To evaluate this integral, let's first multiply it out a little:

$$\begin{aligned} 4\pi\int_{-3}^{3}(4-x)\sqrt{9-x^2}\,dx &= 4\pi\left[\int_{-3}^{3} 4\sqrt{9-x^2}\,dx - \int_{-3}^{3} -x\sqrt{9-x^2}\,dx\right] \\ &= 16\pi\int_{-3}^{3}\sqrt{9-x^2}\,dx + 4\pi\int_{-3}^{3} x\sqrt{9-x^2}\,dx \end{aligned}$$

Now, we have two integrals to evaluate. We don't know an antiderivative of $\sqrt{9-x^2}$, but $y = \sqrt{9-x^2}$ is just the graph of the top half of the circle in our picture. So, $\int_{-3}^{3}\sqrt{9-x^2}$ is the area of the top half of the circle, and we know that a circle of radius 3 has area 9π. So, $\int_{-3}^{3}\sqrt{9-x^2}\,dx = \frac{9\pi}{2}$.

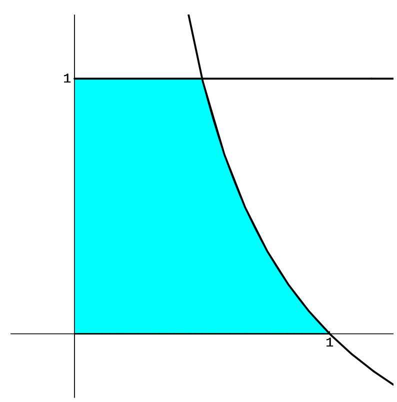
The second integral $\int_{-3}^{3} x\sqrt{9-x^2}\,dx$ can be done using substitution: let $u = 9 - x^2$. Then $du = -2x\,dx$. So, we can rewrite the integrand as $\int -\sqrt{u}\frac{du}{2}$. Since x goes from -3 to 3, u goes from 0 to 0, and $\int_{0}^{0} -\frac{\sqrt{u}}{2}\,du = 0$.

So, our final answer is $16\pi \cdot \frac{9\pi}{2} = \boxed{72\pi^2}$.

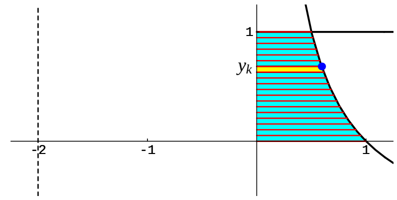
4. Let $\mathcal{R}$ be the region enclosed by the x-axis, the y-axis, $y = 1$, and $y = \frac{1}{x} - 1$.

   (a) *Find the volume generated when $\mathcal{R}$ is rotated about the line $x = -2$.*

   Solution. Here is the region $\mathcal{R}$.



As always, we should first decide whether we want to use vertical or horizontal slices. In this case, if we use vertical slices, we will have to split the region up into where $x < \frac{1}{2}$ and $x > \frac{1}{2}$ because the slices will have different descriptions. Instead, let's use horizontal slices.



Since we're using horizontal slices, we should describe things in terms of y, so let's solve $y = \frac{1}{x} - 1$ for x in terms of y.

$$\begin{aligned} y &= \frac{1}{x} - 1 \\ y + 1 &= \frac{1}{x} \\ \frac{1}{y+1} &= x \end{aligned}$$

So, the blue point is $\left(\frac{1}{y_k+1}, y_k\right)$.

After we rotate our horizontal slice, we will end up with something that is approximately a washer (or CD) with inner radius 2, outer radius $\frac{1}{y_k+1} + 2$, and thickness Δy. So, the volume of the k-th slice is approximately $\left[\pi\left(\frac{1}{y_k+1} + 2\right)^2 - \pi \cdot 2^2\right] \Delta y$. Adding these up and taking the limit gives the integral $\int_0^1 \left[\pi\left(\frac{1}{y+1} + 2\right)^2 - \pi \cdot 2^2\right] dy$. To compute this, we'll first simplify:

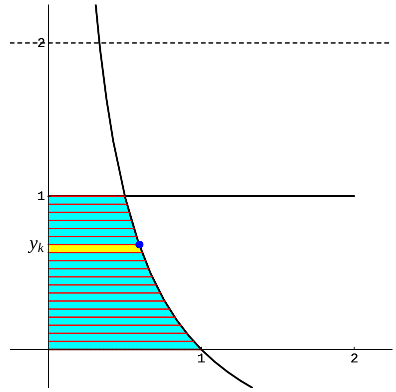
$$\begin{aligned} \int_0^1 \left[\pi\left(\frac{1}{y+1} + 2\right)^2 - \pi \cdot 2^2\right] dy &= \pi \int_0^1 \left[\left(\frac{1}{y+1} + 2\right)^2 - 2^2\right] dy \\ &= \pi \int_0^1 \left[\frac{1}{(y+1)^2} + \frac{4}{y+1} + 4 - 4\right] dy \\ &= \pi \int_0^1 \left[\frac{1}{(y+1)^2} + \frac{4}{y+1}\right] dy \end{aligned}$$

To evaluate, we'll use the substitution $u = y + 1$. Then, $du = dy$ and u goes from 1 to 2 (since y went from 0 to 1). So, we have

$$\begin{aligned} \text{volume} &= \pi \int_1^2 \left(\frac{1}{u^2} + \frac{4}{u}\right) du \\ &= \pi\left(-u^{-1} + 4\ln|u|\right)\Big|_1^2 \\ &= \pi\left[\left(-\frac{1}{2} + 4\ln 2\right) - (-1)\right] \\ &= \boxed{\pi\left(\frac{1}{2} + 4\ln 2\right)} \end{aligned}$$

(b) *Find the volume generated when $\mathcal{R}$ is rotated about the line $y = 2$.*

Solution. For the same reason as in part (a), we'll use horizontal slices.



As we figured out in part (a), the blue point has coordinates $\left(\frac{1}{y_k+1}, y_k\right)$.

After we rotate our horizontal slice, we will end up with a paper towel tube with radius $2-y_k$, height $\frac{1}{y_k+1}$, and thickness Δy. So, the volume of this slice is approximately $2\pi(2-y_k)\frac{1}{y_k+1}\Delta y = 2\pi\frac{2-y_k}{y_k+1}$. Adding these up and taking the limit gives $\lim_{n\to\infty}\sum_{k=1}^{n} 2\pi\frac{2-y_k}{y_k+1}$, which is just the integral $\int_0^1 2\pi\frac{2-y}{y+1}\,dy$.

Now, we have to actually evaluate the integral. Let's try using $u = y+1$. Then, $du = dy$. Also, $y = u-1$, so $2-y = 2-(u-1) = 3-u$. Since y goes from 0 to 1, u goes from 1 to 2. So,

$$\begin{aligned}
\int_0^1 2\pi\frac{2-y}{y+1}\,dy &= 2\pi\int_0^1 \frac{2-y}{y+1}\,dy \\
&= 2\pi\int_1^2 \frac{3-u}{u}\,du \\
&= 2\pi\int_1^2 \left(\frac{3}{u}-1\right)\,du \\
&= 2\pi\,(3\ln|u|-u)\big|_1^2 \\
&= 2\pi[(3\ln 2-2)-(3\ln 1-1)] \\
&= \boxed{2\pi(3\ln 2-1)}
\end{aligned}$$

Integration by Parts

Evaluate the following integrals.

1. $\int xe^x \; dx.$

2. $\int x \ln x \; dx.$

3. $\int_1^e \ln x \; dx.$

4. $\int_0^1 \arctan x \; dx.$

5. $\int x^2 \cos 2x \; dx.$

6. $\int e^x \cos x \; dx.$

7. $\int \cos \sqrt{x} \; dx.$

8. You are given the following information about an unknown function $g(x)$:

$$\int_1^2 \frac{g(u)}{u} \; du = 3, \int_1^2 g(u) \; du = 4, \int_1^4 g(u) \; du = 5, g(1) = 2, g(2) = -2.$$

(a) Evaluate $\int_1^2 (\ln x) g'(x) \; dx.$

(b) Evaluate $\int_1^2 x g(x^2) \; dx.$

9. $\int \sin 5x \sin 3x \; dx.$

Partial Fractions

1. Which of the following is easiest to integrate?

   (a) $\int \frac{5x-4}{x^2-x-2}\,dx.$

   (b) $\int \frac{5x-4}{(x-2)(x+1)}\,dx.$

   (c) $\int \frac{5x}{x^2-x-2}\,dx - \int \frac{4}{x^2-x-2}\,dx.$

   (d) $\int \frac{3}{x+1}\,dx + \int \frac{2}{x-2}\,dx$

   How do the four choices relate to each other?

2. Evaluate the following integrals.

   (a) $\int \frac{1}{y^2-4}\,dy.$

   (b) $\int \frac{5x-7}{x^2-3x+2}\,dx.$

3. Write down the form of the partial fraction expansion for the following integrals. (You don't need to actually solve for the coefficients.)

(a) $\int \frac{3x^2 + x + 5}{(x+1)(x+3)(x-5)}$.

(b) $\int \frac{x+1}{(x^2+4)(x^2+9)}$.

(c) $\int \frac{x^3 + 2x}{(x+4)(x+3)(x+2)^2}$.

(d) $\int \frac{x^2+1}{(x+1)^2(x^2+5)}$.

(e) $\int \frac{3x^2}{x^2+2x+1}\, dx$.

(f) $\int \frac{x^3 + 4x^2 + 7x}{x^2+4x+3}\, dx$.

4. Evaluate the following integrals.

(a) $\int \frac{x^2 - x + 4}{x^3 + 4x}$.

(b) $\int \frac{1}{x^3 + x}\, dx$.

(c) $\int \frac{\sin\theta \; d\theta}{\cos^2\theta + \cos\theta - 2}$.

Integration Techniques

In each problem, decide which method of integration you would use. If you would use substitution, what would u be? If you would use integration by parts, what would u and dv be? If you would use partial fractions, what would the partial fraction expansion look like? (Don't solve for the coefficients.)

1. $\displaystyle\int \frac{\cos x\ dx}{\sqrt{1+\sin x}}.$

   Solution. Use the substitution $u = 1 + \sin x$ since $du = \cos x\ dx$ also appears in the integrand.

2. $\displaystyle\int (\ln x)^2\ dx.$

   Solution. Partial fractions is definitely not right, since this is not a rational function. Substitution doesn't look so promising, so we're left with integration by parts. Since we don't know how to integrate anything involving ln, use $u = (\ln x)^2$ and $dv = dx$.

3. $\displaystyle\int e^x \sin x\ dx.$

   Solution. This is a classic integration by parts integral, where you do integration by parts twice to get back the original integral and then solve for it. You can use $u = e^x$ and $dv = \sin x\ dx$ or $u = \sin x$ and $dv = e^x\ dx$; they work equally well.

4. $\displaystyle\int \frac{x}{x^2-1}\ dx.$

   Solution. Since the integrand is a rational function, you could use partial fractions. But it's easier to just use substitution with $u = x^2 - 1$.

5. $\displaystyle\int x e^{x^2}\ dx.$

   Solution. Substitution with $u = x^2$ since $du = 2x\ dx$ also appears.

6. $\displaystyle\int \frac{x^2}{x^2+4x+3}\ dx.$

   Solution. The integrand is a rational function, and we can factor the denominator pretty easily, so partial fractions is a good choice. Since the integrand is an improper fraction, we start by rewriting it: $\frac{x^2}{x^2+4x+3} = \frac{x^2+4x+3-4x-3}{x^2+4x+3} = 1 - \frac{4x+3}{x^2+4x+3}$. The denominator factors as $(x+1)(x+3)$, so the partial fraction expansion has the form $1 + \frac{A}{x+1} + \frac{B}{x+3}$.

7. $\displaystyle\int \frac{e^t}{1+e^t}\ dt.$

   Solution. Use substitution with $u = 1 + e^t$ since $du = e^t\ dt$ also appears.

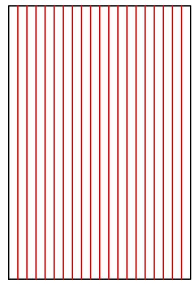
8. $\displaystyle\int \arcsin x\ dx.$

Solution. Partial fractions is definitely out, and there's not much to substitute, so use integration by parts with $u = \arcsin x$ and $dv = dx$.

1. A seaside village, Playa del Carmen, is in the shape of a rectangle 4 miles wide and 6 miles long. The sea lies along a 6-mile long side. People prefer to live near the water, so the density of people is given by $\rho(x) = 10000 - 800x$ people per square mile, where x is the distance from the seaside. We would like to find the population of the village.

   (a) *Show in a sketch how to slice up the region.*

   Solution. We'll slice the region parallel to the sea, like this:



   The reason we do it this way is in part (c).

   (b) *What is the area of the k-th slice?*

   Solution. We'll call the width of each slice Δx. Then, the area of the k-th slice is $6\Delta x$.

   (c) *What is the approximate population in the k-th slice?*

   Solution. This is really the key to the problem. Because of the way we sliced, the population density within each slice is close to being constant. Therefore, we can approximate the population as (population density) times (area), or $\rho(x_k) \cdot 6\Delta x = 6\rho(x_k)\Delta x$.

   (d) *Write a general Riemann sum to estimate the total population of the city.*

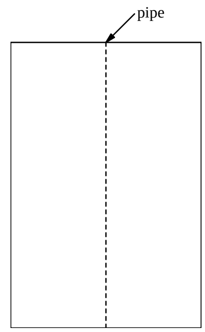
   Solution. To approximate the total population, we just add up the approximate population in all slices, which gives $\sum_{k=1}^{n} 6\rho(x_k)\Delta x$.

   (e) *Find a definite integral expressing the population of the village.*

   Solution. The assumption we made in our approximation was that the population density in each slice was constant. This assumption becomes more accurate as we use more slices, since each slice will be thinner. Therefore, to get the actual population, we should take the limit as $n \to \infty$. We know that $\lim_{n\to\infty} \sum_{k=1}^{n} 6\rho(x_k)\Delta x$ is the same as the integral $\boxed{\int_0^4 6\rho(x)\, dx.}$

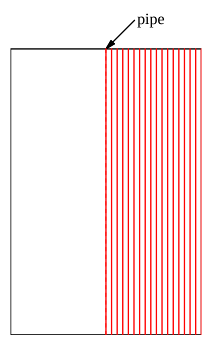
2. *A rectangular plot of farm land is 300 meters by 200 meters. A straight irrigation pipe 300 meters long runs down the center of the plot, dividing it in half lengthwise. The farmer's yield decreases as the distance from the irrigation pipe increases. Suppose that the yield is given by $\rho(x)$ grams per square meter, where x is the distance in meters from the irrigation pipe. Write an integral giving the total yield from the plot.*

   Solution. Here's a sketch.



First, because of symmetry, the yield in the left half of the plot is the same as the yield in the right half. Therefore, we only need to figure out the yield in one half; let's do the right half.

Since the yield depends on the distance from the irrigation pipe, we want to slices in which the distance to the irrigation pipe is approximately constant. Therefore, we'll slice parallel to the pipe:

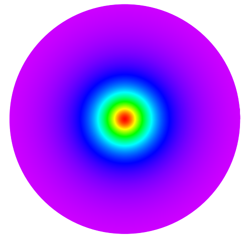


Since x represents the distance to the pipe, the leftmost slice starts where $x = 0$, and the rightmost slice ends where $x = 100$.

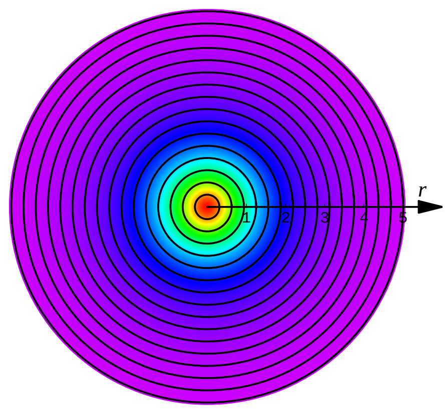
The area of each slice is $300\Delta x$. In the k-th slice, the yield rate is approximately $\rho(x_k)$ grams per square meter, so the yield is approximately $\rho(x_k) \cdot 300\Delta x = 300\rho(x_k)\Delta x$. Summing over all n slices and taking the limit as $n \to \infty$ gives us $\int_0^{100} 300\rho(x)\, dx$ as the yield for half of the plot. Therefore, the total yield is $\boxed{2\int_0^{100} 300\rho(x)\, dx}$.

3. *People in the Boston area like to live near the city center, so the population density around Boston is* $\rho(r) = \frac{36{,}000}{r^2+2r+1}$ *people per square mile, where* r *is the distance in miles to the center of Boston. Find the number of people who live within 5 miles of the center of Boston.*

   Solution. The region we are talking about (within 5 miles of the center of Boston) is a disk, with the highest population density at the center. Here is a plot showing the population density in the city (red represents the highest density, and purple represents the lowest):



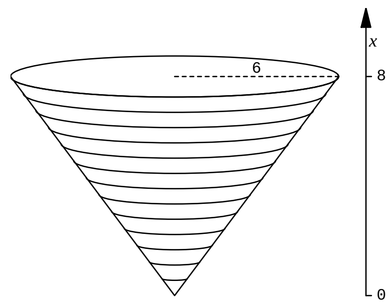
Just like in the other problems, we want to slice the region so that the population density in each slice is almost constant. In this case, we can accomplish that by slicing into concentric rings.



The population density in the k-th slice is approximately equal to $\rho(r_k)$. The area of slice k is approximately $2\pi r_k \Delta r$. So, the number of people living in the k-th slice is approximately $\rho(r_k) \cdot 2\pi r_k \Delta r = 2\pi r_k \rho(r)\Delta$. Summing over all n slices and taking the limit as $n \to \infty$ gives the integral $\boxed{\int_0^5 2\pi r \rho(r)\ dr}$.

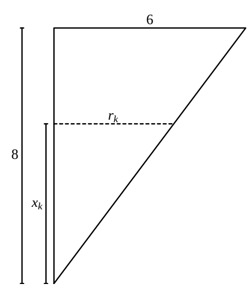
1. *A cone with height 8 inches and radius 6 inches is filled with flavored slush. When the cup is held upright with the pointed end resting on a table, the density of flavoring syrup in the cup varies with height above the table. Suppose $\rho(x)$ gives the number of ounces of syrup per cubic inch, where x is the distance from the table top. Write an integral giving the total amount of syrup in the cup.*

   Solution. Since the density varies with height, we will slice the cone like this:



   The k-th slice looks approximately like a disk of thickness Δx. To approximate the volume of the slice, we need to find its radius. Moreover, we should do this in terms of x_k since we're going to want to end up with an integral in terms of x.

   To find the radius r_k of the k-th slice in terms of x_k, we use similar triangles:



   This shows us that $\frac{6}{8} = \frac{r_k}{x_k}$, so $r_k = \frac{6}{8}x_k = \frac{3}{4}x_k$. Therefore, the volume of the k-th slice is approximately $\pi\left(\frac{3}{4}x_k\right)^2 \Delta x = \frac{9\pi}{16}x_k^2\Delta x$ cubic inches. The amount of syrup in the k-th slice is approximately $[\rho(x_k)\text{ounces per cubic inch}] \cdot \left[\frac{9\pi}{16}x_k^2\Delta x\text{cubic inches}\right] = \frac{9\pi}{16}x_k^2\rho(x_k)\Delta x$ ounces. Summing the amount in each slice gives the Riemann sum approximation $\displaystyle\sum_{k=1}^{n}\frac{9\pi}{16}x_k^2\rho(x_k)\Delta x$. Taking the limit as $n \to \infty$ gives the integral $\boxed{\displaystyle\int_0^8 \frac{9\pi}{16}x^2\rho(x)\,dx}$.

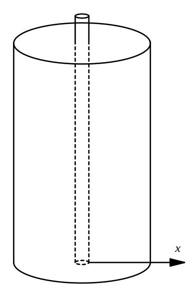
2. *Suppose the density of a planet is given by the function $\rho(r) = \dfrac{40000}{1 + 0.0001r^3}$ kilograms per cubic kilometer, where r is the distance in kilometers from the center of the planet. Find the total mass of the planet if its radius is 8000 km. (You do not need to evaluate your integral.)*

Solution. Since the density varies with the distance to the center, we should slice into concentric spherical shells. Each shell will have a small thickness Δr, and a good approximation for the volume of such a spherical shell is its surface area multiplied by the thickness Δr.

The k-th slice has outer radius r_k and inner radius r_{k-1}, so its volume is approximately $4\pi r_k^2 \Delta r$ km^3. The mass of this slice is approximately $[\rho(r_k)\text{kg} / \text{km}^3] \cdot [4\pi r_k^2 \Delta r \text{ km}^3] = 4\pi r_k^2 \rho(r_k)\Delta r$ kg. Adding these up gives the Riemann sum approximation $\sum_{k=1}^{n} 4\pi r_k^2 \rho(r_k)\Delta r$. Taking the limit gives the integral $\boxed{\int_0^{8000} 4\pi r^2 \rho(r)\, dr}$. If we wanted a numeric answer, we could integrate using the substitution $u = 1 + 0.0001r^3$.

3. *A cylindrical candle of height* 50 *mm and radius* 12 *mm is formed by repeatedly dipping a wick of radius* 1 *mm into hot wax and then allowing the new layer of wax to dry. The density of each new layer is slightly different, so the density of the candle varies with the distance to the wick. If $\rho(x)$ gives the density in grams per cubic mm of the wax, where x measures the distance to the wick, write an integral giving the mass of the candle.*

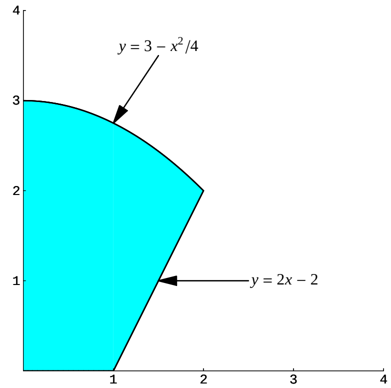
   Solution. Here is a (crude) sketch of our candle.



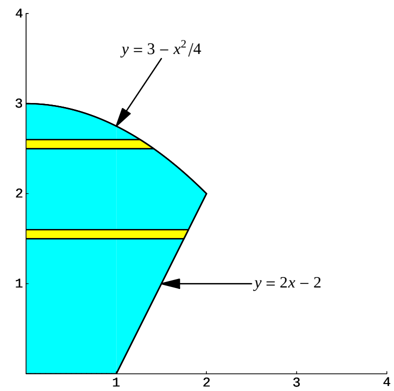
   We should slice using cylindrical shells (also known as paper towel tubes, just like what we used in some of the volumes of revolution problems).

   Each cylindrical shell has height 50 mm and thickness Δx. The k-th shell is distance x_k from the wick, so its radius is $x_k + 1$ (because we must take into account the radius of the wick). So, the volume of the k-th shell is approximately $2\pi(x_k + 1) \cdot 50 \cdot \Delta x = 100\pi(x_k + 1)\Delta x$ cubic inches. Then, its mass is approximately $[\rho(x_k)$ grams per cubic mm$] \cdot [100\pi(x_k + 1)\Delta x$ cubic mm$] = 100\pi(x_k + 1)\rho(x_k)\Delta x$ grams. Adding these up gives the Riemann sum approximation $\sum_{k=1}^{n} 100\pi(x_k + 1)\rho(x_k)\Delta x$. Taking the limit as $n \to \infty$ gives the definite integral $\boxed{\int_0^{11} 100\pi(x + 1)\rho(x)\, dx}$.

4. *We can model a muffin as a solid of revolution, obtained by rotating the following region about the y-axis. Due to a poor recipe, the chocolate chips in our muffin tend to sink to the bottom. The amount of chocolate in the muffin is given by $\rho(y) = 5 - y$ grams per cubic inch, where y represents the distance to the bottom of the muffin. Find the total amount of chocolate in the muffin.*

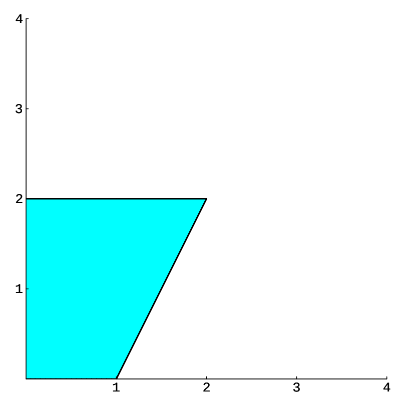


Solution. Since the chocolate density varies with the distance to the bottom of the muffin, we must slice parallel to the bottom of the muffin. Each slice is approximately a disk. Here are two representative slices:



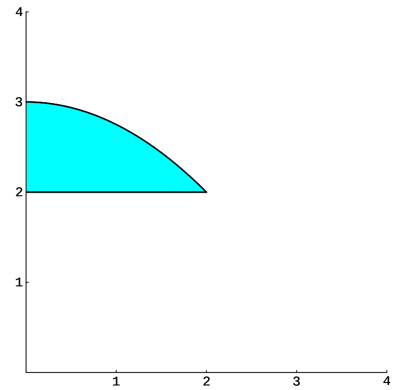
Notice that these two slices have different descriptions: the top one should be described using the curve $y = 3 - \frac{x^2}{4}$, while the bottom one should be described using the curve $y = 2x - 2$. So, we should really consider the top and bottom parts of the muffin separately.

Let's first focus on the bottom part of the muffin, which we get by rotating this region:



Here, the slices are disks, with the radius being the horizontal distance between the y-axis and $y = 2x - 2$. To find this horizontal distance, we need to solve $y = 2x - 2$ for x, which gives $x = \frac{y+2}{2}$. Therefore, the radius of the k-th slice is $\frac{y_k+2}{2}$. This means that its volume is approximately $\pi\left(\frac{y_k+2}{2}\right)^2\Delta y$, so the amount of chocolate in this slice is approximately $\rho(y_k)\cdot\pi\left(\frac{y_k+2}{2}\right)^2\Delta y = \pi\left(\frac{y_k+2}{2}\right)^2\rho(y_k)\Delta y$. Adding these up and taking the limit gives us the integral $\int_0^2 \pi\left(\frac{y+2}{2}\right)^2\rho(y)\,dy$ for the amount of chocolate in the bottom part of the muffin.

To deal with the top part of the muffin, we use exactly the same reasoning. We get the top part of the muffin by rotating this region:



We need to solve $y = 3 - \frac{x^2}{4}$ for x:

$$\begin{aligned} y &= 3 - \frac{x^2}{4} \\ \frac{x^2}{4} &= 3 - y \\ x^2 &= 4(3-y) \\ x &= 2\sqrt{3-y} \end{aligned}$$

So, the k-th slice is approximately a disk of radius $2\sqrt{3-y_k}$ and thickness Δy, which means its volume is approximately $\pi(2\sqrt{3-y_k})^2\Delta y = 4\pi(3-y_k)\Delta y$. Therefore, the amount of chocolate in this slice is approximately $4\pi(3-y_k)\rho(y_k)\Delta y$. Adding these up and taking the limit gives us the integral $\int_2^3 4\pi(3-y)\rho(y)\,dy$ for the amount of chocolate in the top part of the muffin.

So, the total amount of chocolate in the muffin is $\boxed{\int_0^2 \pi\left(\frac{y+2}{2}\right)^2\rho(y)\,dy + \int_2^3 4\pi(3-y)\rho(y)\,dy.}$

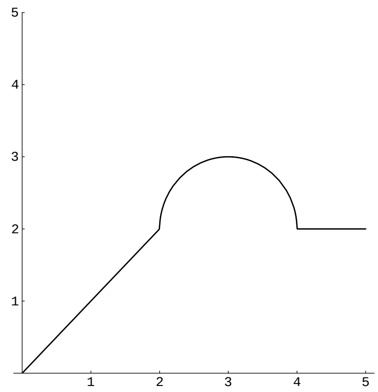
5. *You ride your bike with velocity* $v(t) = 3t^2 + 4t - 5$ *in the time interval* $[0,3]$. *What is your average velocity?*

Solution. It is $\boxed{\frac{1}{3-0}\int_0^3 v(t)\,dt} = \frac{1}{3}\int_0^3 (3t^2+4t-5)\,dt = \frac{1}{3}(t^3+2t^2-5t)\Big|_0^3 = \boxed{10}$.

6. *The temperature outside is given by the function* $f(t)$, *where* t *represents the time since 10:00 am. How would you find the average temperature between noon and 5:00 pm?*

Solution. We want the average temperature for the interval $[2, 7]$, and that's $\boxed{\dfrac{1}{7-2}\displaystyle\int_2^7 f(t)\ dt}$.

7. *The graph of a function f is shown. The graph is made up of lines and semicircles. Find the average value of f on the interval* $[1, 5]$.



Solution. The average value of f is $\dfrac{1}{4}\displaystyle\int_1^5 f(t)\ dt$. We know that $\int_1^5 f(t)\ dt$ is the signed area of f from $t = 1$ to $t = 5$, and from the graph, we can see that this signed area is $\frac{9+\pi}{2}$. Therefore, the average value of the function is $\boxed{\frac{9+\pi}{8}}$.

Arc Length and Improper Integrals

1. *Write an integral that gives the length of one arch of the sine curve (so from $x = 0$ to $x = \pi$).*

   Solution. Our formula tells us that it is $\int_0^\pi \sqrt{1 + \cos^2 x}\, dx$.

2. (a) *Does* $\int_1^\infty \frac{1}{x^2}\, dx$ *converge or diverge? If it converges, evaluate it.*

   Solution. We know that $\int_1^\infty \frac{1}{x^2}\, dx$ really means $\lim_{b\to\infty} \int_1^b \frac{1}{x^2}\, dx$. We can evaluate $\int_1^b \frac{1}{x^2}\, dx$ pretty easily: it is $-\frac{1}{x}\Big|_1^b = -\frac{1}{b} + 1 = 1 - \frac{1}{b}$. So, $\int_1^\infty \frac{1}{x^2}\, dx = \lim_{b\to\infty} 1 - \frac{1}{b} = \boxed{1}$.

   (b) *Does* $\int_1^\infty \frac{1}{x}\, dx$ *converge or diverge? If it converges, evaluate it.*

   Solution. We know that $\int_1^\infty \frac{1}{x}\, dx$ really means $\lim_{b\to\infty} \int_1^b \frac{1}{x}\, dx$, so

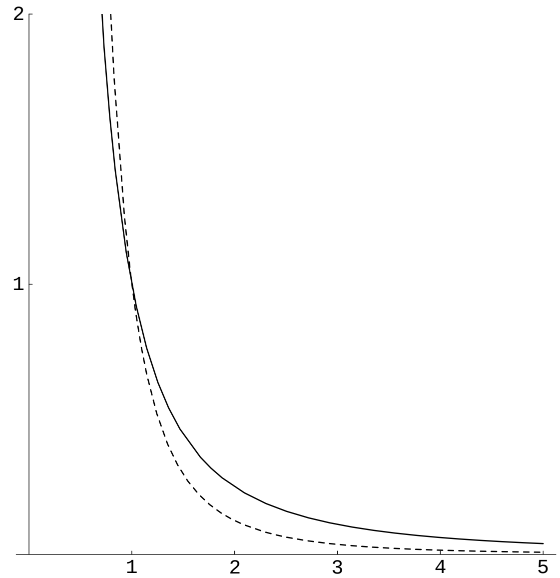
$$\begin{aligned} \int_1^\infty \frac{1}{x}\, dx &= \lim_{b\to\infty} \left(\ln|x|\Big|_1^b \right) \\ &= \lim_{b\to\infty} (\ln|b| - \ln 1) \\ &= \lim_{b\to\infty} \ln|b| \text{ since } \ln 1 = 0 \end{aligned}$$

   But we know that $\lim_{b\to\infty} \ln|b| = \infty$, which is a form of diverging, so the improper integral $\int_1^\infty \frac{1}{x}\, dx$ diverges.

3. Using #2, can you conclude anything about whether the following integrals converge or diverge? (Try to figure this out without evaluating the integrals!)

   (a) $\int_1^\infty \frac{1}{x^3}\, dx$*?*

   Solution. Let's graph $\frac{1}{x^2}$ (the solid curve) and $\frac{1}{x^3}$ (the dashed curve):

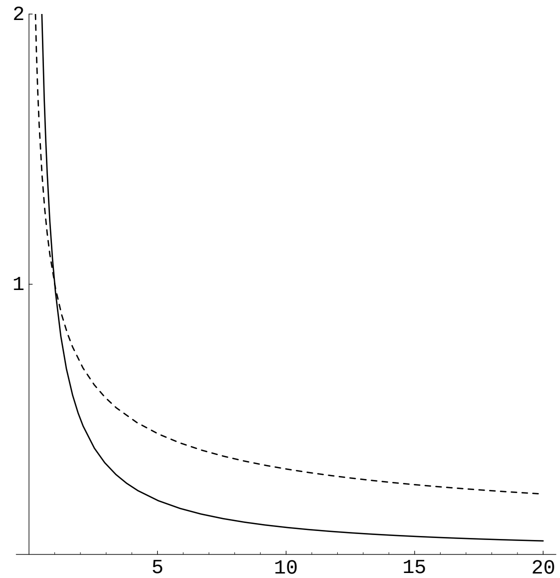


By 2(a), we know that $\int_1^\infty \frac{1}{x^2}\,dx = 1$. Graphically, we interpret this as the area under the curve $y = \frac{1}{x^2}$ to the right of $x = 1$. It is apparent from our picture that the area under $y = \frac{1}{x^3}$ to the right of $x = 1$ should be less than that, so we expect the integral $\int_1^\infty \frac{1}{x^3}\,dx$ to converge. In fact, it does:

$$\begin{aligned}
\int_1^\infty \frac{1}{x^3}\,dx &= \lim_{b\to\infty} \int_1^b \frac{1}{x^3}\,dx \\
&= \lim_{b\to\infty} \left(-\frac{1}{2}x^{-2}\Big|_1^b \right) \\
&= \lim_{b\to\infty} \left(-\frac{1}{2b^2} + \frac{1}{2} \right) \\
&= \frac{1}{2}
\end{aligned}$$

(b) $\int_1^\infty \frac{1}{x^{1/2}}\,dx$?

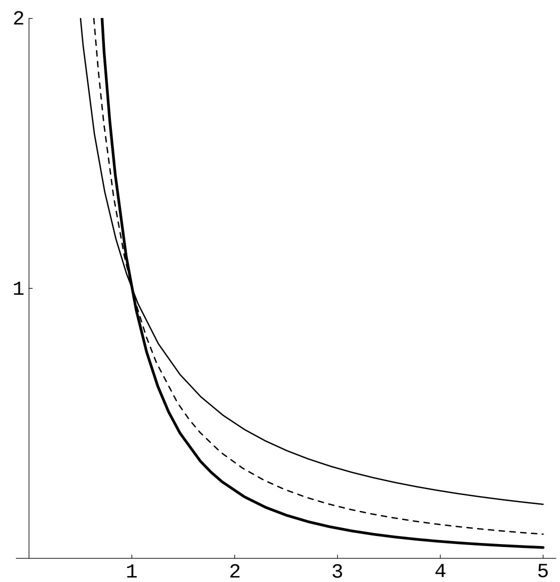
Solution. Let's graph $\frac{1}{x}$ (the solid curve) and $\frac{1}{x^{1/2}}$ (the dashed curve):



Since the graph of $\frac{1}{x^{1/2}}$ is higher than the graph of $\frac{1}{x}$ when $x \geq 1$, the area under $\frac{1}{x^{1/2}}$ to the right of $x = 1$ should be at least as big as the area under $\frac{1}{x}$ to the right of $x = 1$. The area under $\frac{1}{x}$ to the right of $x = 1$ was already infinite, so we expect $\displaystyle\int_1^\infty \frac{1}{x^{1/2}}\, dx$ to diverge.

(c) $\displaystyle\int_1^\infty \frac{1}{x^{3/2}}\, dx$?

Solution. The graph of $\frac{1}{x^{3/2}}$ lies between the graphs of $\frac{1}{x}$ and $\frac{1}{x^2}$:



Here, the thin solid graph is $y = \frac{1}{x}$, the thick solid graph is $y = \frac{1}{x^2}$, and the dashed graph is $y = \frac{1}{x^{3/2}}$. Thinking in terms of areas, we can guess that $\displaystyle\int_1^\infty \frac{1}{x^2}\, dx \leq \int_1^\infty \frac{1}{x^{3/2}}\, dx \leq \int_1^\infty \frac{1}{x}\, dx$.

Using our result from #2, this tells us that $1 \leq \int_1^\infty \frac{1}{x^{3/2}}\ dx \leq \infty$. Unfortunately, that doesn't give us enough information to determine whether $\int_1^\infty \frac{1}{x^{3/2}}\ dx$ converges.

In this case, comparing to $\frac{1}{x}$ and $\frac{1}{x^2}$ doesn't help, so we'll evaluate:

$$\begin{aligned}
\int_1^\infty \frac{1}{x^{3/2}}\ dx &= \lim_{b\to\infty} \int_1^b \frac{1}{x^{3/2}}\ dx \\
&= \lim_{b\to\infty} \left(-2x^{-1/2} \Big|_1^b \right) \\
&= \lim_{b\to\infty} -2b^{-1/2} + 2 \\
&= \lim_{b\to\infty} 2 - \frac{2}{\sqrt{b}} \\
&= 2
\end{aligned}$$

So, the improper integral converges to 2.

Improper Integrals

Determine whether the following integrals converge or diverge. Explain your reasoning.

1. $\displaystyle\int_{-2}^{2} \frac{x}{x^2-1}\,dx.$

   Solution. The integrand is discontinuous at $x = \pm 1$, so we know we need to split the integral. The improprieties are at -1 and 1, and each of our pieces should have at most one impropriety. So, let's split like this:

   $$\int_{-2}^{2} \frac{x}{x^2-1}\,dx = \int_{-2}^{-1} \frac{x}{x^2-1}\,dx + \int_{-1}^{0} \frac{x}{x^2-1}\,dx + \int_{0}^{1} \frac{x}{x^2-1}\,dx + \int_{1}^{2} \frac{x}{x^2-1}\,dx.$$

   (You could choose a number other than 0 between -1 and 1.)

   Now, we have to evaluate each of the integrals on the right (and they are all improper). Let's first find an antiderivative of $\frac{x}{x^2-1}$ by substituting $u = x^2 - 1$:

   $$\int \frac{x}{x^2-1}\,dx = \frac{1}{2}\int \frac{1}{u}\,du = \frac{1}{2}\ln|u| = \frac{1}{2}\ln|x^2-1|.$$

   Now, we start evaluating our four improper integrals using limits.

   $$\begin{aligned}\int_{-2}^{-1} \frac{x}{x^2-1}\,dx &= \lim_{b\to -1^-} \int_{-2}^{b} \frac{x}{x^2-1}\,dx \\ &= \lim_{b\to -1^-} \frac{1}{2}\ln|x^2-1|\Big|_{-2}^{b} \\ &= \lim_{b\to -1^-} \left(\frac{1}{2}\ln|b^2-1| - \frac{1}{2}\ln 3\right)\end{aligned}$$

   As $b \to -1^-$, $b^2 - 1 \to 0$, so $\ln|b^2-1| \to -\infty$. Thus, this integral diverges.

   Since one of our four pieces diverges, we don't need to bother calculating the other pieces; we already know that the whole integral $\boxed{\text{diverges}}$.

2. $\displaystyle\int_{1}^{\infty} \frac{1}{x^4+2}\,dx.$

   Solution. The integrand here is very similar to $\frac{1}{x^4}$, and we know $\displaystyle\int_1^\infty \frac{1}{x^4}\,dx$ converges. This suggests that we use the Comparison Theorem.

   Notice that $0 \le \frac{1}{x^4+2} \le \frac{1}{x^4}$ for all x. Since $\displaystyle\int_1^\infty \frac{1}{x^4}\,dx$ converges, the Comparison Theorem tells us that $\displaystyle\int_1^\infty \frac{1}{x^4+2}\,dx$ also $\boxed{\text{converges}}$.

3. $\displaystyle\int_{0}^{\infty} \frac{1}{e^x+x}\,dx.$

Solution. Since we don't know how to find an antiderivative of $\frac{1}{e^x+x}$, we should use the Comparison Theorem. Since $0 \le \frac{1}{e^x+x} \le \frac{1}{e^x}$ for all x and you saw on your homework that $\displaystyle\int_0^\infty \frac{1}{e^x}\,dx$ converges, the Comparison Theorem tells us that $\displaystyle\int_0^\infty \frac{1}{e^x+x}\,dx$ also $\boxed{\text{converges}}$.

4. $\displaystyle\int_{-\infty}^{\infty} \sin x\;dx.$

Solution. We need to split up this integral because it has two improprieties: the $-\infty$ and the ∞. It doesn't really matter where we split it, so let's split it at 0: $\displaystyle\int_{-\infty}^{\infty} \sin x\;dx = \int_{-\infty}^{0} \sin x\;dx + \int_0^\infty \sin x\;dx$. Let's do $\displaystyle\int_0^\infty \sin x\;dx$ first. By definition, this is $\displaystyle\lim_{b\to\infty}\int_0^b \sin x\;dx = \lim_{b\to\infty} -\cos x|_0^b = \lim_{b\to\infty}(-\cos b+1)$, which does not exist. Since this piece diverges, we know that the whole integral $\boxed{\text{diverges}}$.

5. $\displaystyle\int_1^\infty \frac{1+e^{-x}}{x}\;dx.$

Solution. We can use the Comparison Theorem: $\frac{1+e^{-x}}{x} \ge \frac{1}{x} \ge 0$ when $x \ge 1$. Since we know that $\displaystyle\int_1^\infty \frac{1}{x}\,dx$ diverges, $\displaystyle\int_1^\infty \frac{1+e^{-x}}{x}\,dx$ must $\boxed{\text{diverge}}$ as well.

6. $\displaystyle\int_1^\infty \frac{\cos^2 x}{x^2}\;dx.$

Solution. We can use the Comparison Theorem: $\frac{1}{x^2} \ge \frac{\cos^2 x}{x^2} \ge 0$. Since we know that $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx$ converges, $\displaystyle\int_1^\infty \frac{\cos^2 x}{x^2}\,dx$ $\boxed{\text{converges}}$ as well.

Probability

Waiting times, shelf-lifes, and equipment failure times are often modeled by exponentially decreasing probability density functions.

1. Suppose $f(t) = 0$ for $t < 0$ and $f(t) = 0.5e^{-ct}$ for $t \geq 0$ is the probability density function for the lifetime of a particular toy (t in years).

   (a) *For what value of c is this a probability density function?*

   Solution. In order for f to be a probability density function, it must satisfy $\int_{-\infty}^{\infty} f(t)\, dt = 1$, or $\int_{-\infty}^{0} f(t)\, dt + \int_{0}^{\infty} f(t)\, dt = 1$. The first integral is 0 since $f(t) = 0$ for $t < 0$.

   If $c = 0$, then $f(t) = 0.5$ for $t \geq 0$, and the improper integral $\int_0^\infty f(t)\, dt$ certainly won't converge. If $c < 0$, then $f(t)$ is a positive increasing function, and again $\int_0^\infty f(t)\, dt$ won't converge. So, we must need to have $c > 0$.

   In this case,

$$\begin{aligned} 0.5\int_0^\infty e^{-ct}\, dt &= 0.5 \lim_{b\to\infty} \int_0^b e^{-ct}\, dt \\ &= 0.5 \lim_{b\to\infty} -\frac{1}{c}e^{-ct}\Big|_0^b \\ &= 0.5 \lim_{b\to\infty} \left(-\frac{1}{c}e^{-cb} + \frac{1}{c}\right) \\ &= \frac{0.5}{c} \end{aligned}$$

   (Notice that, in the last step, we needed to use the fact that $c > 0$.) We want this to equal 1, so c should equal $\boxed{0.5}$.

   (b) *What is the probability that the toy lasts over one year? (Is there any way to compute this without computing an improper integral?)*

   Solution. The probability that the toy lasts over one year is given by the integral $\int_1^\infty f(t)\, dt$. If we want to avoid using an improper integral, we could instead calculate the probability that the toy lasts less than one year, which is $\int_0^1 f(t)\, dt$. Then, the probability that the toy lasts over one year is $1 - \int_0^1 f(t)\, dt$. Whichever method you use, the answer is $\boxed{\frac{1}{\sqrt{e}}}$.

   (c) *What is the median life of this type of toy?*

Solution. The median is the value T such that $\int_{-\infty}^{T} f(t)\, dt = \frac{1}{2}$. So, we want

$$\begin{aligned} \frac{1}{2} &= \int_{0}^{T} 0.5e^{-0.5t}\, dt \\ &= -e^{-0.5t}\Big|_{0}^{T} \\ &= -e^{-0.5T} + 1 \end{aligned}$$

Solving, $T = \boxed{-2\ln\frac{1}{2}}$ (this is approximately 1.37 years).

2. *A large number of students take an exam. 30% of the students receive a score of 70, 50% receive a score of 80, and 20% receive a score of 90. What is the average score on the exam?*

   Solution. If the number of students is N, then $.3N$ people scored 70, $.5N$ scored 80, and $.2N$ scored 90. So, the sum of all scores is $.3N(70) + .5N(80) + .2N(90) = 79N$. The average score is this sum divided by the number of people, or $\boxed{79}$.

3. The density function for the duration of telephone calls within a certain city is $p(x) = 0.4e^{-0.4x}$ where x denotes the duration in minutes of a randomly selected call.

   (a) *What percentage of calls last one minute or less?*

   Solution. We are interested in the fraction of calls that last between 0 and 1 minute, which is $\int_{0}^{1} 0.4e^{-0.4x}\, dx = -e^{-0.4x}\Big|_{0}^{1} = 1 - e^{-0.4}$. (As a percent, it is $100(1 - e^{-0.4}) \approx 33\%$.)

   (b) *What percentage of calls last between one and two minutes?*

   Solution. This is $\int_{1}^{2} 0.4e^{-0.4x}\, dx = -e^{-0.4x}\Big|_{1}^{2} = e^{-0.4} - e^{-0.8}$, which is approximately 22%.

   (c) *What percentage of calls last 3 minutes or more?*

   Solution. This is

$$\begin{aligned} \int_{3}^{\infty} 0.4e^{-0.4x}\, dx &= \lim_{b\to\infty} \int_{3}^{b} 0.4e^{-0.4x}\, dx \\ &= \lim_{b\to\infty} -e^{-0.4x}\Big|_{3}^{b} \\ &= \lim_{b\to\infty} e^{-1.2} - e^{-0.4b} \\ &= e^{-1.2} \end{aligned}$$

   This works out to approximately 30%.

   (d) *What is the average length of a call?*

   Solution. The average length of a call is $\int_{0}^{\infty} x(0.4e^{-0.4x})\, dx$. Using integration by parts, an antiderivative of $x(0.4e^{-0.4x})$ is $-xe^{-0.4x} - 2.5e^{-0.4x}$. So,

$$\begin{aligned} \int_{0}^{\infty} x(0.4e^{-0.4x})\, dx &= \lim_{b\to\infty} \left(-xe^{-0.4x} - 2.5e^{-0.4x}\right)\Big|_{0}^{b} \\ &= \lim_{b\to\infty} -be^{-0.4b} - 2.5e^{-0.4b} + 2.5 \qquad (1) \end{aligned}$$

The middle term, $-2.5e^{-0.4b}$, tends to 0 as $b \to \infty$. For the first term, $-be^{-0.4b}$, we need to use L'Hospital's Rule (since the limit is of the form $\infty \cdot 0$):

$$\begin{aligned} \lim_{b\to\infty} -be^{-0.4b} &= \lim_{b\to\infty} -\frac{b}{e^{0.4b}} \\ &= \lim_{b\to\infty} -\frac{1}{0.4e^{0.4b}} \\ &= 0 \end{aligned}$$

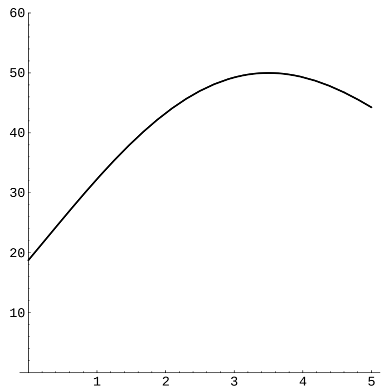
Plugging this into (**??**), the average length of a call is $\boxed{\text{2.5 minutes}}$.

4. *The lifetime, in hundreds of hours, of a certain type of light bulb has been found empirically to have a probability density function approximated by* $f(x) = \frac{\sqrt{65}}{8(1+x^2)^{3/2}}$ *for* $0 < x < 8$. *Find the mean lifetime of such a bulb.*

Solution. The mean lifetime is $\int_0^8 xf(x)\,dx = \frac{\sqrt{65}}{8}\int_0^8 \frac{x}{(1+x^2)^{3/2}}\,dx$. To evaluate this, we substitute $u = 1 + x^2$:

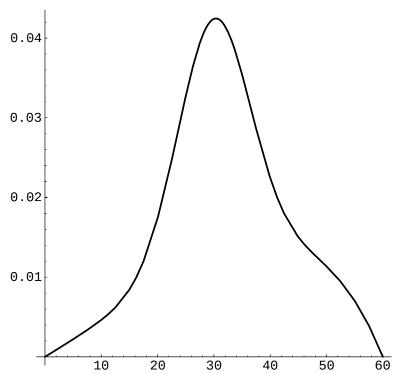
$$\begin{aligned} \frac{\sqrt{65}}{8}\int_0^8 \frac{x}{(1+x^2)^{3/2}}\,dx &= \frac{\sqrt{65}}{8}\int_1^{65} \frac{1}{2}u^{-3/2}\,du \\ &= -\frac{\sqrt{65}}{8}u^{-1/2}\Big|_1^{65} \\ &= \frac{\sqrt{65}}{8}\left(1 - \frac{1}{\sqrt{65}}\right) \\ &= \boxed{\frac{\sqrt{65}-1}{8}} \end{aligned}$$

1. (a) *The following function represents the temperature outside as a function of time. Estimate the average temperature between time* 0 *and time* 5.



Solution. We are looking for the average value of the temperature function, or the average height of its graph. In this graph, it looks like this is around 40. (The exact answer is around 41.5.)

(b) *A meteorologist takes several temperature readings which are described by the following probability density function. Estimate the average temperature.*



Solution. The probability density function has a peak at 30, but the area under the curve to the left is smaller than the area under the curve to the right. This tells us that more of the readings were above 30 than below 30, so we can guess that the average temperature was a bit above 30. It does not look like the average temperature will be as high as 40 because the fraction of temperature readings above 40 (the area under the graph to the right of 40) is pretty small. So, we guess that the average temperature is above 30 but below 40. (The actual value is about 32.)

2. Let $p(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$, a probability density function.

(a) *Sketch the graph of this probability density function. What do you think its mean is?*

Solution. It looks like 0 since the graph is symmetric about $x = 0$.

(b) *Verify your guess mathematically.*

Solution. To find the mean, we need to evaluate $\int_{-\infty}^{\infty} x \cdot \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx$. Since the bounds $-\infty$ and ∞ are both improprieties, we need to split the integral into two pieces and evaluate:

$$\int_{-\infty}^{\infty} x \cdot \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx = \frac{1}{\sqrt{2\pi}} \left(\int_{-\infty}^{0} xe^{-x^2/2}\, dx + \int_{0}^{\infty} xe^{-x^2/2}\, dx \right) \tag{1}$$

Let's first find an antiderivative of $xe^{-x^2/2}$. We use the substitution $u = -x^2/2$ to get

$$\int xe^{-x^2/2}\ x = \int -e^u\ du = -e^u + C = -e^{-x^2/2} + C.$$

Now, we'll go back to (**??**). Let's do the integral from 0 to infinity first. Since it's an improper integral, we really need to take a limit:

$$\begin{aligned} \int_0^{\infty} xe^{-x^2/2}\ dx &= \lim_{b\to\infty} \int_0^b xe^{-x^2/2}\ dx \\ &= \lim_{b\to\infty} -e^{-x^2/2}\Big|_0^b \\ &= \lim_{b\to\infty} 1 - e^{-b^2/2} \\ &= 1 \end{aligned}$$

Similarly, we find that $\int_{-\infty}^{0} xe^{-x^2/2}\ dx = -1$, so (**??**) tells us that the mean is $-1 + 1 = \boxed{0}$.

3. *The bell curve with mean* 0 *and standard deviation* s *is given by the probability density function* $p(x) = \frac{1}{s\sqrt{2\pi}} e^{-x^2/(2s^2)}$. *What fraction of the population is within one standard deviation* s *of the mean 0?*

   Solution. We are looking for the fraction of the population that is between $-s$ and s, which is given by the integral $\boxed{\frac{1}{s\sqrt{2\pi}} \int_{-s}^{s} e^{-x^2/(2s^2)}\ dx}$.

1. *Find the degree 6 Taylor polynomial approximation for $f(x) = \sin x$ centered at 0.*

   Solution. We are looking for a polynomial $P_6(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6$ such that the k-th derivative $P_6^{(k)}(0)$ is equal to the k-th derivative $f^{(k)}(0)$. The k-th derivative $P_6^{(k)}(0)$ is just equal to $k!a_k$, so we want $a_k = \frac{f^{(k)}(0)}{k!}$. The derivatives of $f(x) = \sin x$ are:

$$\begin{array}{rcl@{\quad}c@{\quad}rcl}
f(x) &=& \sin x & \Rightarrow & f(0) &=& 0 \\
f'(x) &=& \cos x & \Rightarrow & f'(0) &=& 1 \\
f''(x) &=& -\sin x & \Rightarrow & f''(0) &=& 0 \\
f'''(x) &=& -\cos x & \Rightarrow & f'''(0) &=& -1 \\
f^{(4)}(x) &=& \sin x & \Rightarrow & f^{(4)}(0) &=& 0 \\
f^{(5)}(x) &=& \cos x & \Rightarrow & f^{(5)}(0) &=& 1 \\
f^{(6)}(x) &=& -\sin x & \Rightarrow & f^{(6)}(0) &=& 0
\end{array}$$

   Therefore, $a_0 = 0$, $a_1 = 1$, $a_2 = 0$, $a_3 = -\frac{1}{3!}$, $a_4 = 0$, $a_5 = \frac{1}{5!}$, and $a_6 = 0$. So, $P_6(x) = \boxed{x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5}$.

2. (a) *If you want to find a Taylor polynomial approximation $a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$ (centered at 0) to $f(x)$, write a formula for the coefficient a_k.*

   Solution. $\boxed{a_k = \frac{f^{(k)}(0)}{k!}}$. (You might wonder what happens when $k = 0$: 0! is defined to be 1, so the formula still works.)

   (b) *If you want to find a Taylor polynomial approximation $a_0 + a_1(x-3) + a_2(x-3)^2 + \cdots + a_n(x-3)^n$ (centered at 3) to $f(x)$, write a formula for the coefficient a_k.*

   Solution. $\boxed{a_k = \frac{f^{(k)}(3)}{k!}.}$

3. *How do you think you would represent $\sin x$ as an infinite polynomial centered at 0? This is called the Taylor series (rather than Taylor polynomial) generated by $\sin x$ about 0.*

   Solution. Based on the pattern we started to see in #1, it seems like we should get

$$\boxed{x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \frac{1}{9!}x^9 - \cdots}.$$

4. *What is the Taylor series generated by $\cos x$ about 0?*

   Solution. Using $f(x) = \cos x$, we have

$$\begin{array}{rcl@{\quad}c@{\quad}rcl}
f(x) &=& \cos x & \Rightarrow & f(0) &=& 1 \\
f'(x) &=& -\sin x & \Rightarrow & f'(0) &=& 0 \\
f''(x) &=& -\cos x & \Rightarrow & f''(0) &=& -1 \\
f'''(x) &=& \sin x & \Rightarrow & f'''(0) &=& 0
\end{array}$$

After this, the derivatives repeat, and we continue to get $1, 0, -1, 0, 1, 0, -1, 0, \ldots$. So, our coefficients are $a_0 = 1$, $a_1 = 0$, $a_2 = -\frac{1}{2}!$, $a_3 = 0$, $a_4 = \frac{1}{4!}$, $a_5 = 0$, and so on. Thus, the Taylor series should be

$$\boxed{1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 - \cdots}.$$

5. *What is the Taylor series generated by e^x about 0?*

   Solution. If $f(x) = e^x$, then the k-th derivative $f^{(k)}(x)$ is always e^x, so $f^{(k)}(0) = 1$. Therefore, the k-th coefficient a_k in the Taylor series is $\frac{1}{k!}$. Thus, the Taylor series is $\boxed{1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots}$.

   If we wanted to write this in summation notation, we would write $\displaystyle\sum_{k=0}^{\infty} \frac{x^k}{k!}$.

6. We hope that, by using "polynomials of infinite degree," we end up with something that is not just an approximation for our function but is actually equal to the function. We don't really know if this is true yet. Taking on faith that e^x is actually equal to its Taylor expansion about 0, can you write a power series expansion (or "infinite polynomial representation") of:

   (a) e^{-x^2}?

   Solution. Since $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$, we can get e^{-x^2} just by replacing all of the x's in the series for e^x with $-x^2$: $e^{-x^2} = \boxed{1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \frac{x^8}{4!} - \cdots}$. In summation notation, $e^{-x^2} = \displaystyle\sum_{k=0}^{\infty} \frac{(-x^2)^k}{k!}$. We often simplify $(-x^2)^k$ as $[(-1)(x^2)]^k = (-1)^k x^{2k}$, so you might also see this as $\displaystyle\sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{k!}$.

   (b) $\displaystyle\int e^{-x^2}\, dx$?

   Solution. If we believe that $e^{-x^2} = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \frac{x^8}{4!} - \cdots$, then it seems plausible that we can integrate this using the reverse of the Power Rule to get $\displaystyle\int e^{-x^2}\, dx = C + x - \frac{1}{3}x^3 + \frac{1}{5}\cdot\frac{x^5}{2!} - \frac{1}{7}\cdot\frac{x^7}{3!} + \frac{1}{9}\cdot\frac{x^9}{4!} - \cdots$, where C is any constant.

7. (a) *Write a general formula for the Taylor series of $f(x)$ centered at 0.*

   Solution. We know it should look like $a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots$ where $a_k = \frac{f^{(k)}(0)}{k!}$. So, it is $\boxed{f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots}$. In summation notation, it is $\displaystyle\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}x^k$.

   (b) *What if you wanted to center at 5?*

   Solution. Then, we would get $a_0 + a_1(x-5) + a_2(x-5)^2 + a_3(x-5)^3 + \cdots$ where $a_k = \frac{f^{(k)}(5)}{k!}$. In other words, we would get $\boxed{f(5) + f'(5)(x-5) + \frac{f''(5)}{2!}(x-5)^2 + \frac{f'''(5)}{3!}(x-5)^3 + \cdots}$.

1. In your quest to become a millionaire by age 50, you start an aggressive savings plan. You open a new investment account on January 1, 2008 and deposit \$9000 into it every year on January 1. Each year, you earn 7% interest on December 31.

   (a) *How much money will you have in your account on January 2, 2009? 2010? 2014? (Don't try to add or multiply things out; just write an arithmetic expression.)*

   Solution. On December 31, 2008, you will have \$9000(1.07) because of the interest you've earned. After you deposit \$9000 on January 1, 2009, you will have $\$9000 + \$9000(1.07)$.

   On December 31, 2009, you will receive your interest, giving you $\$9000(1.07)+\$9000(1.07)^2$. After you make your yearly deposit on January 1, 2010, you will have $\$9000+\$9000(1.07)+\$9000(1.07)^2$.

   Continuing this reasoning, you will have $\$9000 + \$9000(1.07) + \$9000(1.07)^2 + \cdots + \$9000(1.07)^6$ on January 2, 2014.

   (b) *Will you be a millionaire by age 50?*

   Solution. The answer will depend on when you were born. Let's say that you were born in 1988. Then, we want to know what has happened by the year 2038. Using the argument of part (a), on January 2, 2038, you will have $\$9000 + \$9000(1.07) + \$9000(1.07)^2 + \cdots + \$9000(1.07)^{30}$. To see whether this is bigger than a million, we want some way to calculate this sum quickly.

   Let's call this amount S:

   $$S = 9000 + 9000(1.07) + 9000(1.07)^2 + 9000(1.07)^3 + \cdots + 9000(1.07)^{29} + 9000(1.07)^{30} \qquad (1)$$

   Notice that if we multiply both sides by 1.07, we get something similar looking on the right side:

   $$1.07S = 9000(1.07) + 9000(1.07)^2 + 9000(1.07)^3 + \cdots + 9000(1.07)^{30} + 9000(1.07)^{31} \qquad (2)$$

   Subtracting (1) from (2), most of the terms on the right side cancel, and we are left with $0.07S = 9000(1.07)^{31} - 9000$, so $S = \frac{9000(1.07)^{31}-9000}{.07} = \$918,657.37$. So, you are not quite a millionaire, but you are close!

2. If you suffer from allergies, your doctor may suggest that you take Claritin once a day. Each Claritin tablet contains 10 mg of loratadine (the active ingredient). Every 24 hours, about 7/8 of the loratadine in the body is eliminated (so 1/8 remains). [1]

   (a) *If you take one Claritin tablet every morning for a week, how much loratadine is in your body right after you take the 3rd tablet? 7th tablet? (Don't try to simplify your computations; just write out an arithmetic expression.)*

   Solution. Immediately after taking the first tablet, you have 10 mg of loratadine in your body. The following morning, only 1/8 of that is left, so you have 10 (1/8) mg in your body. You then take another pill containing 10 mg, so you have a total of $10 + 10(1/8)$ mg of loratadine in your body.

   The following morning, 1/8 of that remains, or $[10 + 10(1/8)](1/8) = 10(1/8) + 10(1/8)^2$. You then take another pill containing 10 mg, so you have a total of $10 + 10(1/8) + 10(1/8)^2$ mg of loratadine in your body after taking the 3rd pill.

[1] This estimate comes from the fact that the average half-life of loratadine is known to be 8 hours.

Continuing this reasoning, you will have $10+10(1/8)+10(1/8)^2+10(1/8)^3+10(1/8)^4+10(1/8)^5+10(1/8)^6$ mg in your body after the 7th pill.

(b) *If you take Claritin for years and years, will the amount of loratadine in your body level off? Or will your bloodstream be pure loratadine?*

Solution. Right after you take the n-th pill, you will have $10 + 10(1/8) + 10(1/8)^2 + 10(1/8)^3 + \cdots + 10(1/8)^{n-1}$ mg of loratadine in your body. Let's call this amount S_n. We are wondering what happens to S_n as n gets very large.

We use the same trick we used in #1(b) to get a closed form expression for S_n. We said that

$$S_n = 10 + 10(1/8) + 10(1/8)^2 + \cdots + 10(1/8)^{n-2} + 10(1/8)^{n-1} \tag{3}$$

Multiplying both sides by 1/8, we get that

$$(1/8)S_n = 10(1/8) + 10(1/8)^2 + 10(1/8)^3 + \cdots + 10(1/8)^{n-1} + 10(1/8)^n \tag{4}$$

If we subtract (4) from (3), most of the terms on the right side cancel, and we are left with $(7/8)S_n = 10 - 10(1/8)^n$. Dividing both sides by 7/8, $S_n = \frac{10-10(1/8)^n}{7/8}$.

Using this expression for S_n, it is easy to see what happens as n gets bigger and bigger: $(1/8)^n$ gets closer and closer to 0, so $\frac{10-10(1/8)^n}{7/8}$ gets closer and closer to $\frac{10}{7/8} = \frac{80}{7}$. Thus, over time, the amount of loratadine in your body gets closer and closer to $\frac{80}{7}$ mg.

3. *For what values of x does the geometric series $1 + x + x^2 + \cdots$ converge?* [2] *If it converges, what does it converge to?*

Solution. This is a geometric series $a + ar + ar^2 + ar^3 + \cdots$ with $a = 1$ and $r = x$. Therefore, we know that it diverges when $|x| \geq 1$. When $|x| < 1$, it converges to $\frac{1}{1-x}$.

4. Which of the following series are geometric?

(a) $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k 2^{k+1}}{3^k}$.

Solution. A geometric series is a series of the form $a + ar + ar^2 + \cdots$. To decide whether the given series is geometric, we want to see if it matches this form. It's helpful to write out the first few terms. When $k = 1$, we have $\frac{(-1)^1 2^2}{3^1} = -\frac{4}{3}$. When $k = 2$, we have $\frac{(-1)^2 2^3}{3^2} = \frac{8}{9}$. When $k = 3$, we have $\frac{(-1)^3 2^4}{3^3} = -\frac{16}{27}$. When $k = 4$, we have $\frac{(-1)^4 2^5}{3^4} = \frac{32}{81}$. So far, it looks like it could be the geometric series $a + ar + ar^2 + ar^3 + \cdots$ with $a = -\frac{4}{3}$ and $r = -\frac{2}{3}$.

To see if this is correct, let's compare the k-th terms. The k-th term of the given series is $\frac{(-1)^k 2^{k+1}}{3^k}$, while the k-th term of the geometric series is ar^{k-1}. So, we are hoping that $\frac{(-1)^k 2^{k+1}}{3^k} = \left(-\frac{4}{3}\right)\left(-\frac{2}{3}\right)^{k-1}$. If you multiply out the right side, you will see that this is indeed the case, so the given series is geometric with $a = -\frac{4}{3}$ and $r = -\frac{2}{3}$.

(b) $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^3}$.

[2] We could also write this series in summation notation as $\displaystyle\sum_{k=0}^{\infty} x^k$.

Solution. If we write this series out, it is $1 + \frac{1}{8} + \frac{1}{27} + \frac{1}{64} + \cdots$. We can already see that it's not geometric because the terms don't have a common ratio. (To elaborate: if it was geometric, the first term would be a and the second term would be ar; this means that a would have to be 1, and r would have to be $\frac{1}{8}$. But then the third term isn't right.)

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{2}{3^{n/2}}$.

Solution. We could also write this series as $\frac{2}{3^{1/2}} + \frac{2}{3} + \frac{2}{3^{3/2}} + \frac{2}{3^2} + \cdots$, which looks like it might be geometric with $a = \frac{2}{3^{1/2}}$ and $r = \frac{1}{3^{1/2}}$.

To check if this is correct, we want to see if the n-th term is $ar^{n-1} = \frac{1}{3^{1/2}} \left(\frac{1}{3^{1/2}}\right)^{n-1}$, and it is. So, the series is geometric with $a = \frac{2}{3^{1/2}}$ and $r = \frac{1}{3^{1/2}}$.

1. Suppose you know that the infinite series $a_1 + a_2 + a_3 + \cdots + a_n + \cdots$ converges to s and that $a_k > 0$ for k any positive integer. Let $s_n = a_1 + a_2 + a_3 + \cdots + a_n$. For each of the following statements, determine whether the statement must be true, could possibly be true, or must be false.

   (a) $\lim_{n\to\infty} a_n = 0$.

   (b) $\lim_{n\to\infty} s_n = 0$.

   (c) There exists a number M such that $s_n < M$ for all n. (This is equivalent to saying that the partial sums are bounded. Why?)

   (d) $\sum_{k=5}^{\infty} a_k$ converges.

   Solution. (a) must be true, (b) must be false, (c) must be true, and (d) must be true. (See the solutions to Homework 15 for more details.)

2. *Suppose you know that* $\lim_{n\to\infty} b_n = 0$. *Can you be sure that the infinite series* $b_1+b_2+b_3+\cdots$ *converges?*

   Solution. No; the harmonic series in #5 is an example of a series that diverges even though its terms tend to 0.

3. (a) *Give an example of a sequence (ordered list) of numbers such that the numbers are increasing but are bounded.*

   Solution. $0.9, 0.99, 0.999, 0.9999, 0.99999, \ldots$ is one such example.

   (b) *Give an example of a sequence (ordered list) of numbers such that the numbers are increasing and are not bounded.*

   Solution. $1, 2, 3, 4, 5, \ldots$.

   (c) *Give an example of a sequence (ordered list) of numbers such that the numbers are bounded but have no limit as* $n \to \infty$.

   Solution. $0, 1, 0, 1, 0, 1, 0, 1, \ldots$.

4. (a) *A sequence which is both monotonic and bounded*

   must converge   *could either converge or diverge*   *must diverge*

   Solution. Must converge. This is the Monotonic Sequence Theorem.

   (b) *A sequence which is monotonic but not bounded*

   must converge   *could either converge or diverge*   *must diverge*

   Solution. Must diverge.

5. Consider the series $\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$ (called the harmonic series).

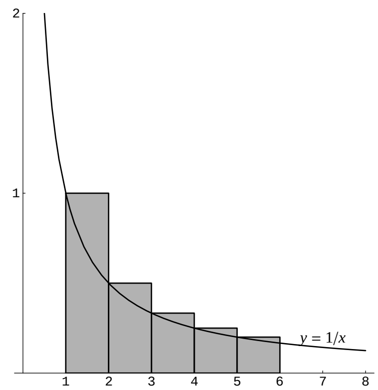
(a) *Does the sequence of terms converge? If so, to what does it converge?*

Solution. Yes, the terms $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$ are getting closer to 0.

(b) *Does the sequence of partial sums converge? If so, to what does it converge?*

Solution. The sequence of partial sums does not converge. The sequence of partial sums is definitely increasing (every partial sum is bigger than the previous partial sum). So, if we knew the sequence of partial sums was not bounded, we could conclude that it didn't converge (by #4(b)).

One way to see that the sequence of partial sums is not bounded is to look at them graphically. The partial sums can be represented as areas. For instance, the 5th partial sum is $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}$, which we can represent as the area in these 5 boxes:



As we can see, the 5th partial sum is therefore bigger than $\int_1^6 \frac{1}{x}\,dx$. In general, the n-th partial sum is bigger than $\int_1^{n+1} \frac{1}{x}\,dx$. Taking the limit as $n \to \infty$ of $\int_1^{n+1} \frac{1}{x}\,dx$ gives us the improper integral $\int_1^{\infty} \frac{1}{x}\,dx$, which we know diverges. So, the sequence of partial sums also diverges.

The other way to understand these partial sums is using a clever trick. Let's look at the partial sums $s_1, s_2, s_4, s_8, s_{16}, \ldots$.

$$
\begin{aligned}
s_1 &= 1 \\
s_2 &= 1+\frac{1}{2} \\
s_4 &= 1+\frac{1}{2}+\left(\frac{1}{3}+\frac{1}{4}\right) \\
&> 1+\frac{1}{2}+\left(\frac{1}{4}+\frac{1}{4}\right) \\
&= 1+\frac{1}{2}+\frac{1}{2} \\
s_8 &= 1+\frac{1}{2}+\left(\frac{1}{3}+\frac{1}{4}\right)+\left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right) \\
&> 1+\frac{1}{2}+\left(\frac{1}{4}+\frac{1}{4}\right)+\left(\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}\right) \\
&= 1+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}
\end{aligned}
$$

$$\begin{aligned} s_{16} &= 1+\frac{1}{2}+\left(\frac{1}{3}+\frac{1}{4}\right)+\left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right)+\left(\frac{1}{9}+\frac{1}{10}+\cdots+\frac{1}{16}\right) \\ &> 1+\frac{1}{2}+\left(\frac{1}{4}+\frac{1}{4}\right)+\left(\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}\right)+\left(\frac{1}{16}+\frac{1}{16}+\cdots+\frac{1}{16}\right) \\ &= 1+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2} \end{aligned}$$

Since this pattern continues, there is no way that the sequence of partial sums can be bounded.

(c) *Does the series* $\sum_{k=1}^{\infty} \frac{1}{k}$ *converge?*

Solution. This is really the same question as (b); the sequence of partial sums diverges, so the series diverges.

(d) *Does the series* $\sum_{k=10^{10}}^{\infty} \frac{1}{100000000k}$ *converge?*

Solution. No. We know that $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges. Therefore, the series $\sum_{k=1}^{\infty} \frac{1}{100000000k}$ diverges (the whole thing has just been multiplied by a constant). We know that the beginning terms of a series do not affect its convergence, so if we start with the 10^{10}-th term, the series still diverges.

6. See if you can determine whether each of the following series converges or diverges by using the Nth Term Test for Divergence, results about geometric series, or some sort of comparison to series you know about.

(a) $\sum_{k=100}^{\infty} \frac{1}{3k}$.

Solution. This is basically the harmonic series multiplied by $\frac{1}{3}$, except that the first 99 terms are missing. Neither of things affects convergence, so this series $\boxed{\text{diverges}}$, just like the harmonic series.

(b) $\sum_{k=3}^{\infty} \frac{(-1)^k 2^k}{3^k}$.

Solution. We can rewrite this as $\sum_{k=3}^{\infty} \left(-\frac{2}{3}\right)^k$, which is a geometric series with common ratio $-\frac{2}{3}$. We know that such a series $\boxed{\text{converges}}$.

(c) $\sum_{k=4}^{\infty} \frac{(-1)^k 3^k}{2^k}$.

Solution. We can rewrite this as $\sum_{k=4}^{\infty} \left(-\frac{3}{2}\right)^k$, which is a geometric series with common ratio $-\frac{3}{2}$. We know that such a series $\boxed{\text{diverges}}$. Alternatively, you could use the Nth Term Test for Divergence: $\lim_{k\to\infty} \frac{(-1)^k 3^k}{2^k}$ does not exist, so the Nth Term Test for Divergence tells us that the series cannot converge either.

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n}$.

Solution. Intuitively, we know that a series converges if its terms go to 0 "quickly enough." The terms of this series do go to 0, but $\frac{\ln n}{n} > \frac{1}{n}$ when $n > e$, so the terms of the given series go to 0 more slowly than the terms of the harmonic series. Since we know that the harmonic series diverges, we should guess that this series diverges also.

To make it more precise mathematically, we could say that the partial sums of this series grow more quickly than the partial sums of the harmonic series. Since the partial sums of the harmonic series already grow without bound, the partial sums of this series also grow without bound, so this series $\boxed{\text{diverges}}$.

(e) $\displaystyle\sum_{n=2}^{\infty} \frac{n}{\ln n}$.

Solution. We can use the Nth Term Test for Divergence: $\displaystyle\lim_{n\to\infty} \frac{n}{\ln n} = \lim_{n\to\infty} \frac{1}{1/n} = \lim_{n\to\infty} n = \infty$ (using L'Hospital's Rule in the first step). Since the terms of this series do not tend to 0, the series $\boxed{\text{diverges}}$.

(f) $\displaystyle\sum_{n=0}^{\infty} \sin n$.

Solution. Again, we can use the Nth Term Test for Divergence: $\displaystyle\lim_{n\to\infty} \sin n$ does not exist, so the series $\boxed{\text{diverges}}$.

Comparison

1. Use the Comparison Test (also known as "direct comparison") to decide whether the following series converge or diverge.

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}3^n}$.

   Solution. When n is really big, 3^n is much much bigger than $\sqrt{n}$, so it seems like the terms of this series are affected more by 3^n than $\sqrt{n}$. Therefore, let's compare to the geometric series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3^n}$. When $n \geq 1$, $0 \leq \frac{1}{\sqrt{n}3^n} \leq \frac{1}{3^n}$. The geometric series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3^n}$ converges (the common ratio is 1/3), so the Comparison Test tells us that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}3^n}$ also converges.

   (b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n!}$.

   Solution. If we "unpack" the summation notation, we get $1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} + \cdots$. These terms seem to go to 0 pretty quickly — certainly more quickly, say, than the geometric series $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$, which we know converges. So, let's compare to the geometric series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^{n-1}}$. (Why is it 2^{n-1} instead of 2^n, you might be wondering? Because we're starting with $n = 1$, but we want the first term in the geometric series to be 1.)

   We'd like to say that $0 \leq \frac{1}{n!} \leq \frac{1}{2^{n-1}}$. The first inequality is obviously true. Now, $\frac{1}{n!}$ means $\frac{1}{n} \cdot \frac{1}{n-1} \cdot \frac{1}{n-2} \cdots \frac{1}{2}\frac{1}{1} = \frac{1}{n} \cdot \frac{1}{n-1} \cdot \frac{1}{n-2} \cdots \frac{1}{2}$, while $\frac{1}{2^{n-1}}$ means $\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdots \frac{1}{2}$. Both products have $n-1$ terms, and all of the terms in the product for $\frac{1}{2^{n-1}}$ are at least as big as the corresponding term in $\frac{1}{n!}$. So, it is indeed the case that $0 \leq \frac{1}{n!} \leq \frac{1}{2^{n-1}}$. Since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^{n-1}}$ converges (it's a geometric series with common ratio 1/2), the Comparison Test tells us that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n!}$ also converges.

   (c) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{n^3 - n + 1000}$.

   Solution. You might have the intuition that the terms $\frac{n^2}{n^3-n+1000}$ "grow like" $\frac{n^2}{n^3} = \frac{1}{n}$, so the series should act like $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ and diverge. To verify this, we should compare to $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$.

   Let's try: we want to say that $0 \leq \frac{1}{n} \leq \frac{n^2}{n^3-n+1000}$. The first inequality is always true, but the second inequality is only true when $n^3 \geq n^3 - n + 1000$, or $n \geq 1000$. This isn't a big problem though; as we know, the beginning terms of a series don't affect convergence.

   So, here's the appropriate reasoning: $0 \leq \frac{1}{n} \leq \frac{n^2}{n^3-n+1000}$ when $n \geq 1000$. We know that $\displaystyle\sum_{n=1000}^{\infty} \frac{1}{n}$ diverges (this is the harmonic series without the first 999 terms, and we know the harmonic series diverges). Therefore, the Comparison Test tells us that $\sum_{n=1000}^{\infty} \frac{n^2}{n^3 - n + 1000}$ also diverges.

Adding in finitely many terms at the beginning doesn't affect convergence, so $\sum_{n=1}^{\infty} \frac{n^2}{n^3 - n + 1000}$ also diverges.

(d) $\sum_{n=1}^{\infty} \frac{1}{\ln(1+n)}$.

Solution. The only series we understand so far are geometric series and the harmonic series. We know that $\ln x$ grows more slowly than x, so it seems like we should compare this to the harmonic series.

By graphing $\ln(1+x)$ and x, we can see that $\ln(1+x) \leq x$ when $x \geq 0$, so $\frac{1}{\ln(1+n)} \geq \frac{1}{n} \geq 0$ for $n \geq 1$. Therefore, the Comparison Test tells us that $\sum_{n=1}^{\infty} \frac{1}{\ln(1+n)}$ diverges.

(To prove that $\ln(1+x) \leq x$, you could do something like this: using what you learned in Math 1a, you can show that the global minimum of $x - \ln(1+x)$ is 0. This means that $x - \ln(1+x) \geq 0$ for all x, so $x \geq \ln(1+x)$.)

2. *True or false: If* $\{a_n\}$ *is a sequence with positive terms and* $\lim_{n\to\infty} a_n = 0$*, then there is a number* k *such that* $a_n < 1$ *whenever* $n \geq k$.

Solution. Roughly, the statement is saying that, if $\{a_n\}$ is a sequence of positive numbers whose limit is 0, then after a while, all of the numbers in the sequence must be less than 1. This is true; after all, if the limit is 0, that means all of the terms in the sequence must stay really close to 0 after a while.

3. Decide whether the following series converge or diverge using any method you like.

(a) $\sum_{n=100}^{\infty} \cos n$.

Solution. We know that a series converges if its terms go to 0 "quickly enough". In this case, the terms aren't going to 0 at all!

To make this precise, $\lim_{n\to\infty} \cos n$ does not exist, so the Nth Term Test for Divergence says that the series will diverge.

(b) $\sum_{k=1}^{\infty} \frac{(-1)^k 2^{k+1}}{3^k}$.

Solution. We can rewrite the k-th term as $\frac{(-1)^k \cdot 2 \cdot 2^k}{3^k} = 2\left(\frac{-2}{3}\right)^k$, so the series is $\sum_{k=1}^{\infty} 2\left(\frac{-2}{3}\right)^k$. This is a geometric series whose first term is $-\frac{4}{3}$ and whose common ratio is $-\frac{2}{3}$. Therefore, we know that this series converges.

(c) $1 + 0 + (-1) + 1 + 0 + (-1) + 1 + 0 + (-1) + \cdots$.

Solution. One way to approach this is simply to write down the sequence of partial sums: $1, 1, 0, 1, 1, 0, 1, 1, 0, \ldots$. Since the sequence of partial sums diverges, the series diverges. (This is using the *definition* of convergence/divergence of a series, not any particular test.)

Alternatively, we could use the Nth Term Test for Divergence: the sequence of terms is $a_1 = 1, a_2 = 0, a_3 = -1, a_4 = 1, a_5 = 0, a_6 = -1, \ldots$. Since $\lim_{n\to\infty} a_n$ does not exist, the Nth Term Test for Divergence says that the series must diverge.

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n}$.

Solution. We can compare this to the harmonic series: $\frac{\ln n}{n} \geq \frac{1}{n} \geq 0$ as long as $n \geq 3$. The series $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n}$ diverges, so the Comparison Test tells us that $\displaystyle\sum_{n=3}^{\infty} \frac{\ln n}{n}$ diverges as well. Adding on finitely many terms at the beginning doesn't change whether a series converges or diverges, so $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n}$ diverges.

More Comparison

1. *For what values of p is the* <u>*p-series*</u> $\sum_{n=1}^{\infty} \frac{1}{n^p}$ *convergent?*

   Solution. If we let $f(x) = \frac{1}{x^p}$, then the terms of our series are just $f(1), f(2), f(3), \ldots$. Also, $f(x)$ is positive, continuous, and decreasing on $[1, \infty)$, so we can use the Integral Test. The Integral Test says $\sum_{n=1}^{\infty} \frac{1}{n^p}$ and $\int_1^{\infty} \frac{1}{x^p}\, dx$ either both converge or both diverge.

   We know that $\int_1^{\infty} \frac{1}{x^p}$ converges when $p > 1$ and diverges when $p \leq 1$. So, the p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges when $p > 1$ and diverges when $p \leq 1$.

2. *Does the series* $\sum_{n=2}^{\infty} \frac{1}{n^2 - 1}$ *converge or diverge?*

   Solution. This looks quite similar to $\sum_{n=2}^{\infty} \frac{1}{n^2}$, but direct comparison doesn't work too well because $\frac{1}{n^2-1} \geq \frac{1}{n^2}$. Instead, we'll use the Limit Comparison Test and compare to $\sum_{n=2}^{\infty} \frac{1}{n^2}$.

   We have $\lim_{n\to\infty} \frac{1/(n^2-1)}{1/n^2} = \lim_{n\to\infty} \frac{n^2}{n^2-1} = 1$. Since $\sum_{n=2}^{\infty} \frac{1}{n^2}$ converges (by #1), the Limit Comparison Test says that $\sum_{n=2}^{\infty} \frac{1}{n^2-1}$ converges.

3. *Does the series* $\sum_{k=1}^{\infty} \frac{3}{8^k - 2}$ *converge or diverge?*

   Solution. When k is large, $\frac{3}{8^k-2} \approx \frac{3}{8^k}$. The series $\sum_{k=1}^{\infty} \frac{3}{8^k}$ is geometric with common ratio $\frac{1}{8}$, so it converges. Therefore, we expect our series to converge as well. However, we can't compare directly because $\frac{3}{8^k-2} \geq \frac{3}{8^k}$. Instead, we'll use the Limit Comparison Test to compare these two series.

   We have $\lim_{k\to\infty} \frac{3/(8^k-2)}{3/8^k} = \lim_{k\to\infty} \frac{8^k}{8^k-2} = 1$. Since $\sum_{k=1}^{\infty} \frac{3}{8^k}$ converges, the Limit Comparison Test says that $\sum_{k=1}^{\infty} \frac{3}{8^k-2}$ converges as well.

4. *Otto is given the following problem for homework.*

   Decide whether the series $\sum_{n=1}^{\infty} \sin^2(\pi n)$ *converges or diverges. Explain your reasoning.*

   Otto writes

The improper integral $\int_1^\infty \sin^2(\pi x)\,dx$ *diverges, so* $\sum_{n=1}^\infty \sin^2(\pi n)$ *also diverges by the Integral Test.*

Otto is correct that the improper integral diverges (although he should have shown more work!). But the rest of his reasoning is incorrect — why? And what is the correct answer to the problem?

Solution. The Integral Test doesn't apply to Otto's problem because the function $\sin^2(\pi x)$ is not a decreasing function. In fact, using the Integral Test gives Otto the wrong answer: $\sin^2(\pi n) = 0$ whenever n is an integer, so the series in Otto's problem is just $0 + 0 + 0 + \cdots$, which converges.

5. Let $\sum a_n$ and $\sum b_n$ be series with positive terms. The Limit Comparison Test only applies when $\lim_{n\to\infty} \frac{a_n}{b_n}$ is a positive real number.

   (a) *If* $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$*, can you draw any conclusions?*

   Solution. Intuitively, the fact that $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ means that a_n goes to 0 a lot more quickly than b_n. So, if $\sum b_n$ converges, then $\sum a_n$ converges.

   In fact, you showed this in Problem Set 17 (#38a from Stewart §8.3).

   If $\sum b_n$ diverges, $\sum a_n$ could still converge; an example is $a_n = \frac{1}{n}$ and $b_n = 1$.

   (b) *If* $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$*, can you draw any conclusions?*

   Solution. Intuitively, the fact that $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$ means that a_n goes to 0 a lot more slowly than b_n. So, if $\sum b_n$ diverges, then $\sum a_n$ diverges as well.

   To make this mathematically correct, we can use the Comparison Test. The fact that $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$ means that, eventually (when n is big enough), $\frac{a_n}{b_n}$ will always be greater than 1. So, eventually, $a_n > b_n$, and the Comparison Test says that $\sum a_n$ will have to diverge since $\sum b_n$ does. (Just like in #1(c) from the "Comparison" handout, it's not true that $a_n > b_n$ for all n, just when n is big enough. But that's fine because we know the beginning terms of a series don't affect whether it converges.)

   If $\sum b_n$ converges, we can't draw any conclusion about $\sum a_n$.

Absolute and Conditional Convergence

1. *Does the series* $\displaystyle\sum_{k=1}^{\infty}(-1)^{k+1}\frac{1}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots$ *converge or diverge? (This series is often called the alternating harmonic series.)*

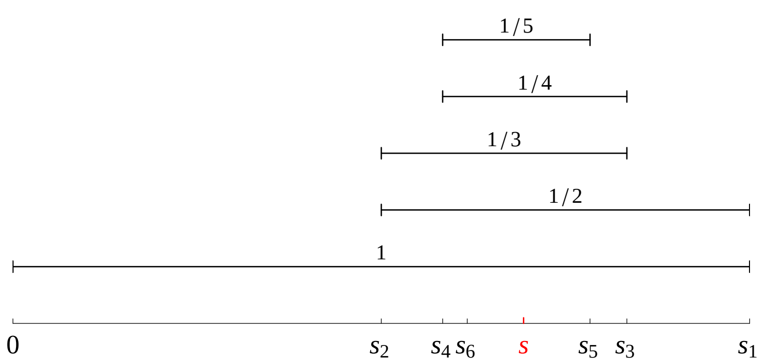
   Solution. To see if the series converges, let's see if we can apply the Alternating Series Test. The terms are decreasing in magnitude: $\frac{1}{k+1} \leq \frac{1}{k}$. In addition, the terms approach 0: $\displaystyle\lim_{k\to\infty}\frac{1}{k} = 0$. Therefore, the Alternating Series Test applies, and we can conclude that $\displaystyle\sum_{k=1}^{\infty}(-1)^{k+1}\frac{1}{k}$ converges.

2. *In fact,* $1 - \dfrac{1}{2} + \dfrac{1}{3} - \dfrac{1}{4} + \dfrac{1}{5} - \cdots = \ln 2$. *Write a finite sum which estimates* $\ln 2$ *with error of less than* 0.001. *Is your approximation too big or too small?*

   Solution. The Alternating Series Estimate tells us that the magnitude of the error in using the n-th partial sum $s_n = \displaystyle\sum_{k=1}^{n}(-1)^{k+1}\frac{1}{k}$ is at most the magnitude of the next term, or $\frac{1}{n+1}$. That is, $|s - s_n| \leq \frac{1}{n+1}$. We want $|s - s_n| < 0.001$, so let's find an n satisfying $\frac{1}{n+1} < 0.001$. (Then we'll have $|s - s_n| \leq \frac{1}{n+1} < 0.001$.)

   To get $\frac{1}{n+1} < 0.001$, we need $n + 1 > 1000$. The smallest n which makes this work is $n = 1000$, so we can use the 1000th partial sum $\boxed{1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots - \frac{1}{999} + \frac{1}{1000}}$.

   To determine whether our approximation is too big or too small, let's go back and look at our diagram of partial sums. Here it is, with the actual sum s in red:



   From the diagram, we can see that the even partial sums $s_2, s_4, s_6, \ldots$ are all $\boxed{\text{too small}}$.

3. *Is the alternating harmonic series* $\displaystyle\sum_{k=1}^{\infty}(-1)^{k+1}\frac{1}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots$ *absolutely convergent?*

Solution. Asking whether the series is absolutely convergent is the same as asking whether the series $\sum_{n=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots$ converges. This is the harmonic series, which diverges. Thus, the series $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k}$ is not absolutely convergent.

4. Determine whether each series converges or diverges. If it converges, does it converge absolutely or conditionally?

   (a) $\sum_{k=1}^{\infty} \frac{(-1)^k}{\sqrt{k}}$.

   Solution. We can apply the Alternating Series Test: $\frac{1}{\sqrt{k+1}} \leq \frac{1}{\sqrt{k}}$ and $\lim_{k\to\infty} \frac{1}{\sqrt{k}} = 0$, so the Alternating Series Test says that $\sum_{k=1}^{\infty} \frac{(-1)^k}{\sqrt{k}}$ converges.

   To decide whether it converges absolutely, we look at the series of absolute values, $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$. This is a p-series with $p = \frac{1}{2}$, and we know that diverges. So, $\sum_{k=1}^{\infty} \frac{(-1)^k}{\sqrt{k}}$ $\boxed{\text{converges conditionally}}$.

   (b) $\sum_{k=1}^{\infty} \frac{\sin k}{k!}$.

   Solution. This series is not an alternating series, so we should not even try the Alternating Series Test. All of our other tests involve series with positive terms, so let's look at the series of absolute values first. That is, we'll first look at $\sum_{k=1}^{\infty} \frac{|\sin k|}{k!}$.

   We can use the Comparison Test here: $0 \leq \frac{|\sin k|}{k!} \leq \frac{1}{k!}$. We know that $\sum_{k=1}^{\infty} \frac{1}{k!}$ converges (see #1(b) from the "Comparison" handout), so $\sum_{k=1}^{\infty} \frac{|\sin k|}{k!}$ converges.

   This means that $\sum_{k=1}^{\infty} \frac{\sin k}{k!}$ $\boxed{\text{converges absolutely}}$. We know that a series which converges absolutely also converges.

   (c) $\sum_{k=1}^{\infty} (-2)^k$.

   Solution. This is an alternating series, but the Alternating Series Test does not apply because the terms are not decreasing in magnitude. That does tell us whether the series converges though.

   Instead, notice that the terms aren't going to 0: $\lim_{k\to\infty} (-2)^k$ does not exist. So, the Nth Term Test for Divergence says that the series $\boxed{\text{diverges}}$. (Alternatively, you could justify by saying that the series is geometric with common ratio -2.)

5. (a) *The Taylor series for* $\cos x$ *about 0 is* $1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{k=0}^{\infty}(-1)^{k+1}\frac{x^{2k}}{(2k)!}$. *Show that, if you plug in any value of* x *with* $-0.5 \le x \le 0.5$, *the series converges.*

Solution. We will show that we can apply the Alternating Series Test. If $-0.5 \le x \le 0.5$, then $\frac{x^{2(k+1)}}{[2(k+1)]!} \le \frac{x^{2k}}{(2k)!}$. This is true because the expression on the left has a bigger denominator and smaller numerator than the expression on the right (in the numerator, x is raised to a higher exponent, and $|x| < 1$). So, the first condition in the Alternating Series Test is satisfied.

Next, we need to show that $\lim_{k\to\infty} \frac{x^{2k}}{(2k)!} = 0$. In fact, $\lim_{k\to\infty} x^{2k} = 0$ (since x is between -0.5 and 0.5), so $\lim_{k\to\infty} \frac{x^{2k}}{(2k)!} = 0$ as well.

Therefore, the Alternating Series Test applies, and we can conclude that the series converges.

(b) *In fact, the series converges for all* x, *and* $\cos x$ *is actually equal to the series; that is,*

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots .$$

Suppose you use the approximation $\cos x \approx 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$ *to approximate* $\cos x$ *when* $-0.5 \le x \le 0.5$. *Find an upper bound for the error. (This means: find a number* U *that you can show is bigger than the error.)*

Solution. The Alternating Series Estimation Theorem tells us that the magnitude of the error is at most that of the first unused term, which is $\frac{x^6}{6!}$. Since $|x| \le 0.5$, the error is at most $\frac{0.5^6}{6!} \approx 0.0000217014$.

Ratio Test

1. What does the Ratio Test tell you about the following series?

   (a) $\displaystyle\sum_{k=0}^{\infty}(-1)^{k+1}\frac{1000^k}{k!}$.

   Solution. Since $\displaystyle\lim_{k\to\infty}\left|\frac{(-1)^{(k+1)+1}\frac{1000^{k+1}}{(k+1)!}}{(-1)^{k+1}\frac{1000^k}{k!}}\right| = \lim_{k\to\infty}\frac{1000^{k+1}\cdot k!}{(k+1)!\cdot 1000^k} = \lim_{k\to\infty}\frac{1000}{k+1} = 0$, the Ratio Test says that the series $\boxed{\text{converges absolutely}}$.

   (b) $\displaystyle\sum_{k=1}^{\infty}\frac{1}{k}$.

   Solution. $\displaystyle\lim_{k\to\infty}\left|\frac{\frac{1}{k+1}}{\frac{1}{k}}\right| = \lim_{k\to\infty}\left|\frac{k}{k+1}\right| = 1$, so the Ratio Test is $\boxed{\text{inconclusive}}$. (Of course, we know the series diverges.)

   (c) $\displaystyle\sum_{k=1}^{\infty}\frac{1}{k^2}$.

   Solution. $\displaystyle\lim_{k\to\infty}\left|\frac{\frac{1}{(k+1)^2}}{\frac{1}{k^2}}\right| = \lim_{k\to\infty}\left|\frac{k^2}{(k+1)^2}\right| = 1$, so the Ratio Test is $\boxed{\text{inconclusive}}$. (Of course, we know the series converges.)

2. *When we studied Taylor series, we found that the Taylor series for* $\sin x$ *about* 0 *was* $x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots$*, which can be written in summation notation as* $\displaystyle\sum_{k=0}^{\infty}(-1)^k\frac{x^{2k+1}}{(2k+1)!}$*. For what values of* x *does this series converge?*

   Solution. We'll use the Ratio Test:

$$\begin{aligned}\lim_{k\to\infty}\left|\frac{(-1)^{k+1}\frac{x^{2(k+1)+1}}{[2(k+1)+1]!}}{(-1)^k\frac{x^{2k+1}}{(2k+1)!}}\right| &= \lim_{k\to\infty}\left|\frac{x^{2k+3}\cdot(2k+1)!}{(2k+3)!\cdot x^{2k+1}}\right| \\ &= \lim_{k\to\infty}\left|\frac{x^2}{(2k+3)(2k+2)}\right| \\ &= 0.\end{aligned}$$

   (Since we're taking the limit as k tends to infinity, we treat x as a constant when taking the limit.) Therefore, the Ratio Test says that, no matter what x is, the series converges absolutely. That is, the series $\boxed{\text{converges absolutely for all } x}$.

3. *When we studied Taylor series, we found that the Taylor series for* $\ln(1+x)$ *about* 0 *was* $x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots$*, which can be written in summation notation as* $\displaystyle\sum_{k=1}^{\infty}(-1)^{k+1}\frac{x^k}{k}$*. For what values of* x *does this series converge?*

Solution. We use the Ratio Test:

$$\begin{aligned}
\lim_{k\to\infty}\left|\frac{(-1)^{(k+1)+1}\frac{x^{k+1}}{k+1}}{(-1)^{k+1}\frac{x^k}{k}}\right| &= \lim_{k\to\infty}\left|\frac{x^{k+1}\cdot k}{(k+1)\cdot x^k}\right| \\
&= \lim_{k\to\infty}\left|x\frac{k}{k+1}\right| \\
&= |x|
\end{aligned}$$

By the Ratio Test, the series converges absolutely when $|x| < 1$ and diverges when $|x| > 1$. However, the Ratio Test is inconclusive when $|x| = 1$, so we'll have to test $x = \pm 1$ separately.

When $x = 1$, the series is $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$. This is the alternating harmonic series, and we've seen that this converges conditionally. (See #1 from the "Absolute and Conditional Convergence" handout.)

When $x = -1$, the series is $-1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{4} - \cdots$, which is just -1 times the harmonic series, and we know that this diverges.

So, our final answer is:

$$\boxed{\sum_{k=1}^{\infty}(-1)^{k+1}\frac{x^k}{k}\quad \begin{cases} \text{converges absolutely} & \text{when } |x| < 1 \\ \text{diverges} & \text{when } |x| > 1 \text{ or } x = -1 \\ \text{converges conditionally} & \text{when } x = 1 \end{cases}}$$

4. Decide whether the following series converge absolutely, converge conditionally, or diverge. You may use any method you like, but explain your reasoning. There is one that you will not be able to do (this is not due to a personal failing; it's just that all of the tests that we know are inconclusive).

   (a) $\displaystyle\sum_{n=1}^{\infty}\frac{\cos n}{n^2}$.

   Solution. When sin or cos appears in a series, it's often helpful to use the comparison test and the fact that $|\cos x|, |\sin x| \leq 1$. Remember that we need a series with positive terms to use the comparison test. So, let's look at the series of absolute values, $\displaystyle\sum_{n=1}^{\infty}\frac{|\cos n|}{n^2}$.

   Since $0 \leq \frac{|\cos n|}{n^2} \leq \frac{1}{n^2}$ and $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges (it's a p-series with $p = 2$), the Comparison Test says that $\displaystyle\sum_{n=1}^{\infty}\frac{|\cos n|}{n^2}$ converges.

   This tells us that $\displaystyle\sum_{n=1}^{\infty}\frac{\cos n}{n^2}$ $\boxed{\text{converges absolutely}}$.

   (b) $\displaystyle\sum_{n=100}^{\infty}\frac{n!n!}{(2n)!}$.

Solution. This has lots of factorials, so the Ratio Test is a good test to try.

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{\frac{(n+1)!(n+1)!}{(2n+2)!}}{\frac{n!n!}{(2n)!}}\right| &= \lim_{n\to\infty}\frac{(n+1)!(n+1)!(2n)!}{n!n!(2n+2)!} \\
&= \lim_{n\to\infty}\frac{(n+1)(n+1)}{(2n+1)(2n+2)} \\
&= \lim_{n\to\infty}\frac{n^2+2n+1}{4n^2+6n+2} \\
&= \frac{1}{4}
\end{aligned}$$

Therefore, the Ratio Test says that the series $\boxed{\text{converges absolutely}}$.

(c) $\displaystyle\sum_{n=0}^{\infty}\frac{\sin n}{n}$.

Solution. None of the tests we know work here.

(d) $\displaystyle\sum_{n=2}^{\infty}\frac{\ln n}{n}$.

Solution. We will use the Comparison Test: $0 \le \frac{1}{n} \le \frac{\ln n}{n}$ when $n > e$. We know that $\displaystyle\sum_{n=3}^{\infty}\frac{1}{n}$ diverges (it's the harmonic series), so the Comparison Test says that $\displaystyle\sum_{n=3}^{\infty}\frac{\ln n}{n}$ diverges, too. Adding on a term at the beginning doesn't affect convergence, so $\displaystyle\sum_{n=2}^{\infty}\frac{\ln n}{n}$ also $\boxed{\text{diverges}}$.

(e) $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n}{n^3+1}$.

Solution. When n is really big, $(-1)^{n+1}\frac{n}{n^3+1} \approx (-1)^{n+1}\frac{n}{n^3} = (-1)^{n+1}\frac{1}{n^2}$. We know that the series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges (it's a p-series with $p = 2$), so we can guess that the given series is probably absolutely convergent.

To verify this, we'll compare the series of absolute values $\displaystyle\sum_{n=1}^{\infty}\frac{n}{n^3+1}$ to $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}$ using the Limit Comparison Test (we are allowed to use the Limit Comparison Test because both of these series have positive terms; however, we may not use the Limit Comparison Test with the original series since it has both positive and negative terms).

Since $\displaystyle\lim_{n\to\infty}\frac{\frac{n}{n^3+1}}{\frac{1}{n^2}} = \lim_{n\to\infty}\frac{n^3}{n^3+1} = 1$ and $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges, $\displaystyle\sum_{n=1}^{\infty}\frac{n}{n^3+1}$ converges by the Limit Comparison Test. This means that $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n}{n^3+1}$ $\boxed{\text{converges absolutely}}$.

(f) $\sum_{n=5000}^{\infty} (-1)^n \frac{n}{n+1}$.

Solution. When n is really big, $(-1)^{n+1}\frac{n}{n+1} \approx (-1)^{n+1}\frac{n}{n} = (-1)^n$. These aren't going to 0, so we should use the Nth Term Test for Divergence: $\lim_{n\to\infty} (-1)^n \frac{n}{n+1}$ does not exist (half of the terms are getting closer to 1 while the other half are getting closer to -1), so the given series $\boxed{\text{diverges}}$.

Power Series

A <u>power series</u> centered at the number a is a series of the form $\sum_{n=0}^{\infty} c_n(x-a)^n$ where x is a variable and the c_n are constants.

1. For what values of x does the power series $\sum_{n=1}^{\infty} n!x^n$ converge? (This series is centered at 0.)

Theorem. For a given power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ centered at a, there are 3 possibilities:

1. The series converges only when $x = a$.
2. The series converges for all x.
3. There is a positive number R such that the series converges when $|x-a| < R$ and diverges when $|x-a| > R$. R is called the <u>radius of convergence</u>. (Note that this doesn't say anything about what happens when $|x-a| = R$.)

The <u>interval of convergence</u> of a power series is the set of x for which the power series converges.

2. Find the radius of convergence and interval of convergence of the power series $\sum_{n=1}^{\infty} \frac{2^n}{n}(x-3)^n$.

3. We know that the power series $\sum_{n=0}^{\infty} x^n$ converges to $\frac{1}{1-x}$ when $|x| < 1$. Find a power series representation of the function $\frac{x}{1+4x^2}$. What is the radius of convergence of this power series?

Theorem. If the power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ has radius of convergence R where $R > 0$ or $R = \infty$, then the function $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$ is differentiable on the interval $(a-R, a+R)$ and

1. $f'(x) = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1}$.

2. $\int f(x)\ dx = C + \sum_{n=0}^{\infty} \frac{c_n}{n+1}(x-a)^{n+1}$.

The power series in (1) and (2) both have radius of convergence R. (Note: Although the radius of convergence remains unchanged, the interval of convergence may change.)

4. (a) Find a power series representation for $\ln(1+x)$ centered at 0. What is the radius of convergence for the power series you have found? (Hint: What is the derivative of $\ln(1+x)$?)

   (b) Find the degree 5 Taylor polynomial approximation of $\ln(1+x)$.

5. Find a power series representation of $\arctan(5x)$ centered at 0. What is the radius of convergence of the power series you have found?

More on Power Series

1. Suppose we have a power series $\sum_{n=1}^{\infty} c_n (x+7)^n$.

   (a) *If you know that the power series converges when $x = 0$, what conclusions can you draw?*

   Solution. The power series is centered at -7, so the fact that it converges at $x = 0$ means that the interval of convergence is at least $(-14, 0]$.

   (b) *Suppose you also know that the power series diverges when $x = 1$. Now what conclusions can you draw?*

   Solution. The interval of convergence is at most $[-15, 1)$.

   (c) *Does $\sum_{n=1}^{\infty} c_n$ converge (assuming that the power series converges when $x = 0$ and diverges when $x = 1$)?*

   Solution. This is the power series when $x + 7 = 1$, or $x = -6$. In part (a), we found that -6 must be in the interval of convergence, so the series $\boxed{\text{converges}}$.

   (d) *Does $\sum_{n=1}^{\infty} c_n (-8.1)^n$ converge?*

   Solution. This is the power series when $x + 7 = -8.1$, or $x = -15.1$. In part (b), we found that -15.1 cannot be in the interval of convergence, so the series $\boxed{\text{diverges}}$.

   (e) *Does $\sum_{n=1}^{\infty} c_n (-8)^n$ converge?*

   Solution. This is the power series when $x+7 = -8$, or $x = -15$. Neither (a) nor (b) tells us what must happen, so there is $\boxed{\text{not enough information}}$ to determine whether the series converges.

2. (a) *Taking for granted that $\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$ for all x, find the Taylor series of $x \sin(x^3)$ at 0.*

   Solution. The theorem tells us that, if we can find a power series representation of $x \sin(x^3)$, then that is the Taylor series. So, rather than trying to find the Taylor series directly (by taking derivatives), let's look for a power series representation of $x \sin(x^3)$.

   We are told that

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \text{ for all } x.$$

   Replacing x by x^3 everywhere gives:

$$\sin x^3 = \sum_{n=0}^{\infty} (-1)^n \frac{x^{6n+3}}{(2n+1)!},$$

still true for all x since the first equation was true for all x. Now, we multiply both sides by x to get

$$x \sin x^3 = \boxed{\sum_{n=0}^{\infty} (-1)^n \frac{x^{6n+4}}{(2n+1)!}},$$

still for all x.

(b) *What is the radius of convergence of the power series you found in part (a)?*

Solution. We said in part (a) that the power series representation was valid for all x, so the radius of convergence of the series must be $\boxed{\infty}$.

(c) *Let $f(x) = x\sin(x^3)$. What is $f'''(0)$? $f^{(4)}(0)$?*

Solution. In part (a), we found that $f(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{6n+4}}{(2n+1)!}$. Unpacking the summation notation, $f(x) = x^4 - \frac{x^{10}}{3!} + \frac{x^{16}}{5!} - \cdots$. Using this fact, there are two ways to solve the problem.

The slick way: The theorem says that, since $f(x) = x^4 - \frac{x^{10}}{3!} + \frac{x^{16}}{5!} - \cdots$, $x^4 - \frac{x^{10}}{3!} + \frac{x^{16}}{5!} - \cdots$ is the Taylor series of $f(x)$ at 0. This means that the coefficient of the x^n term is $\frac{f^{(n)}(0)}{n!}$ (because this is the formula we used to find the coefficients of the Taylor series).

So, $\frac{f'''(0)}{3!} = 0$ (because 0 is the coefficient of the x^3 term in $x^4 - \frac{x^{10}}{3!} + \frac{x^{16}}{5!} - \cdots$) and $\frac{f^{(4)}(0)}{4!} = 1$ (because 1 is the coefficient of the x^4 term in $x^4 - \frac{x^{10}}{3!} + \frac{x^{16}}{5!} - \cdots$). Solving, we get $\boxed{f'''(0) = 0}$ and $\boxed{f^{(4)}(0) = 4!}$.

A slower, but equally valid method: Another way you can tackle this problem is to start again with $f(x) = x^4 - \frac{x^{10}}{3!} + \frac{x^{16}}{5!} - \cdots$ and just differentiate. We get:

$$\begin{aligned}
f(x) &= x^4 - \frac{x^{10}}{3!} + \frac{x^{16}}{5!} - \cdots \\
f'(x) &= 4x^3 - \frac{10x^9}{3!} + \frac{16x^{15}}{5!} - \cdots \\
f''(x) &= 4 \cdot 3x^2 - \frac{10 \cdot 9x^8}{3!} + \frac{16 \cdot 15x^{14}}{5!} - \cdots \\
f'''(x) &= 4 \cdot 3 \cdot 2x - \frac{10 \cdot 9 \cdot 8x^7}{3!} + \frac{16 \cdot 15 \cdot 14x^{13}}{5!} - \cdots \\
f^{(4)}(x) &= 4 \cdot 3 \cdot 2 - \frac{10 \cdot 9 \cdot 8 \cdot 7x^6}{3!} + \frac{16 \cdot 15 \cdot 14 \cdot 13x^{12}}{5!} - \cdots
\end{aligned}$$

If we plug in $x = 0$ to the last two equations, we get $f'''(0) = 0$ and $f^{(4)}(0) = 4 \cdot 3 \cdot 2$.

Note: Either of these two methods should be a lot faster than just starting with $f(x) = x\sin(x^3)$ and differentiating that 4 times; this is one advantage of being able to write $f(x)$ as a power series. Of course, if you wanted to know $f^{(100)}(0)$, the first method is going to be a lot faster than the second.

3. (a) *Find a power series representation of* $\arctan(5x)$ *centered at* 0.

Solution. Let's start by finding a power series representation of $\arctan x$. Then we can replace x by $5x$ to get a power series representation of $\arctan(5x)$.

We know that the derivative of $\arctan x$ is $\frac{1}{1+x^2}$.

We start with:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \text{ valid when } |x| < 1.$$

Let's replace x by $-x^2$:

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-x^2)^n, \text{ valid when } |-x^2| < 1.$$

We'll simplify the right side a little:

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n}, \text{ valid when } |-x^2| < 1.$$

Note that $|-x^2| = |x^2| = |x|^2$, so saying $|-x^2| < 1$ is the same as saying $|x|^2 < 1$, or $|x| < 1$. That is, the radius of convergence of the power series on the right is 1.

Now integrate both sides:

$$\arctan x = C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}.$$

Integrating a power series doesn't change the radius of convergence, so the radius of convergence of this power series is still 1.

We need to solve for the constant of integration C; we do this by plugging in $x = 0$ on both sides of the equation:

$$\arctan 0 = C + 0,$$

so $C = 0$. Therefore,

$$\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}.$$

Since the radius of convergence of this power series is 1, the power series converges when $|x| < 1$ and diverges when $|x| > 1$.

Finally, we replace x by $5x$:

$$\arctan 5x = \sum_{n=0}^{\infty} (-1)^n \frac{(5x)^{2n+1}}{2n+1}.$$

The power series converges when $|5x| < 1$ and diverges when $|5x| > 1$. It's nice to simplify this a little bit, so we end up with

$$\arctan 5x = \boxed{\sum_{n=0}^{\infty} (-1)^n \frac{5^{2n+1}}{2n+1} x^{2n+1}}.$$

(b) *What is the radius of convergence of the power series you found in part (a)?*

Solution. In part (a), we said that the power series converges when $|5x| < 1$ and diverges when $|5x| > 1$. In other words, the power series converges when $|x| < \frac{1}{5}$ and diverges when $|x| > \frac{1}{5}$, so the radius of convergence is $\boxed{\frac{1}{5}}$.

4. In each part, find a power series that has the given interval of convergence. (Hint: If you get stuck, try finding the interval of convergence of $\sum_{n=1}^{\infty} \frac{x^n}{n}$.)

(a) $(-6, 0)$.

Solution. We know the geometric series $\sum_{k=0}^{\infty} x^k$ converges when $|x| < 1$ and diverges when $|x| \geq 1$. We're looking for something that converges when $|x+3| < 3$ and diverges when $|x+3| \geq 3$. Another way of saying this is that we want something that converges when $\left|\frac{x+3}{3}\right| < 1$ and diverges when $\left|\frac{x+3}{3}\right| \geq 1$. The geometric series $\boxed{\sum_{k=0}^{\infty} \left(\frac{x+3}{3}\right)^k}$ works. (Of course, there are infinitely many other possible answers.)

(b) $(-1, 3)$.

Solution. Now we want something that converges when $\left|\frac{x+1}{2}\right| < 1$ and diverges when $\left|\frac{x+1}{2}\right| \geq 1$. One possibility is the geometric series $\boxed{\sum_{k=0}^{\infty} \left(\frac{x+1}{2}\right)^k}$.

(c) *Challenge:* $[-1, 3)$.

Solution. This is more difficult because we can't use a geometric series. (The interval of convergence of a geometric series never includes its endpoints, but here we want to include the left endpoint.) Rememeber that the times we've had series where one endpoint is included but the other is not is when one endpoint gives us the alternating harmonic series (convergent) and the other gives us the harmonic series (divergent). So, let's try to make a power series where plugging in $x = -1$ gives us the alternating harmonic series and plugging in $x = 3$ gives us the harmonic series.

Using the hint, we'll look at $\sum_{n=1}^{\infty} \frac{x^n}{n}$. This has an interval of convergence of $[-1, 1)$, so it's almost what we want. Let's try to stretch and translate the function so that its interval of converge will be $[-1, 3)$. First, we want to stretch it so that the radius of convergence is 2 instead of 1. To do this, we replace x by $\frac{x}{2}$: $\sum_{n=1}^{\infty} \frac{(x/2)^n}{n}$. Next, we want to shift the center to 1: to do this, we replace x by $x - 1$: $\sum_{n=1}^{\infty} \frac{((x-1)/2)^n}{n}$, or $\boxed{\sum_{n=1}^{\infty} \frac{(x-1)^n}{2^n \cdot n}}$.

Series Problems

1. Decide whether the following series converge or diverge. Explain your reasoning.

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{2n^5 + 500n^4 + n^3}{n^7 + 200n^6}$.

   (b) $\displaystyle\sum_{n=100}^{\infty} \frac{\sin n}{n^2}$.

2. Use a second degree Taylor polynomial to approximate $33^{1/5}$.

3. We define a function $f(x)$ by setting $f(x) = \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{\sqrt{n} 2^n}$ for those x for which the series converges.

(a) Find the radius of convergence.

(b) Write a power series representation of $f'(x)$, the first derivative of f. Use it to find a series for $f'(1)$.

(c) Write out the first three non-zero terms of your series for $f'(1)$. At $x = 1$, is the function f increasing or decreasing? Explain.

Differential Equations: An Introduction to Modeling

In #1 - #8, write a differential equation that reflects the situation. Include an initial condition if the information is given.

1. *The population of a certain country increases at a rate proportional to the population size. Let $P = P(t)$ be the population at time t.*

   Solution. The rate of change is $\frac{dP}{dt}$, and we also know that the rate of change is proportional to P, so it's kP for some k. (We know k must be positive because the population is increasing.) So, $\boxed{\frac{dP}{dt} = kP}$.

2. *A snowball melts at a rate proportional to its surface area. At time 0, the snowball has a radius of 10 cm. Let $r = r(t)$ be the radius of the snowball at time t.*

   Solution. The first sentence can be written as $\boxed{\frac{dr}{dt} = k(4\pi r^2)}$, where k is a constant. (In this case, k is going to be negative because the rate of change should be negative.) The second sentence can be written as $\boxed{r(0) = 10}$.

3. *A yellow rubber duck is dropped out of the window of an apartment building at a height of 80 feet. Let $s = s(t)$ be the height of the duck above the ground at time t. (Gravity is the acceleration -32 ft/s^2.)*

   Solution. The rubber duck accelerates due to gravity, so $\boxed{s''(t) = -32}$. We know that it starts at 80 feet above the ground, so $\boxed{s(0) = 80}$. We also know that the duck is not moving at the beginning, so its initial velocity is 0. That is, $\boxed{s'(0) = 0}$.

4. *Ferdinand is trying to fill a bucket from a faucet. Unfortunately, he doesn't realize that there is a small hole in the bottom of the bucket. Water flows in to the bucket from the faucet at a constant rate of .75 quarts per minute, and it flows out of the hole at a rate proportional to the amount of water $W(t)$ already in the bucket (due to the increased water pressure).*

   Solution. The rate of change of water in the bucket is $\frac{dW}{dt}$, which is equal to (rate of water coming in) minus (rate of water coming out). The rate of water coming in is 0.75. The rate of water going out is $kW(t)$ where k is a positive constant. So, $\boxed{\frac{dW}{dt} = 0.75 - kW}$.

5. *A drug is being administered to a patient at a constant rate of c mg/hr. The patient metabolizes and eliminates the drug at a rate proportional to the amount in his body. Let $M = M(t)$ be the amount (in mg) of medicine in the patient's body at time t, where t is measured in hours.*

   Solution. The rate of change of medicine in the patient's body is equal to (rate in) minus (rate out). The rate in is c. The rate out is proportional to the amount in his body, so it's kM for some positive constant k. Therefore, the appropriate model is $\boxed{\frac{dM}{dt} = c - kM}$.

6. *\$6000 is deposited in a bank account. The account has a nominal annual interest rate of 2%, compounded continuously. There are no deposits and no withdrawals. Let $M = M(t)$ be the amount of money in the account at time t, where t is measured in years.*

Solution. If $t = 0$ is the time the money is deposited, then the first sentence is saying that $\boxed{M(0) = 6000}$. The second sentence is saying that $\boxed{\frac{dM}{dt} = .02M}$.

7. *\$6000 is deposited in a bank account. The account has a nominal annual interest rate of 2%, compounded continuously. Money is being withdrawn at a rate of \$500 per year.*[1] *Let $M = M(t)$ be the amount of money in the account at time t, where t is measured in years.*

   Solution. The rate at which money is leaving the account is 500. Everything else is the same as in the previous problem, so we have $\boxed{\frac{dM}{dt} = .02M - 500}$ and $\boxed{M(0) = 6000}$.

8. *A rumor spreads at a rate proportional to product of the number of people who have heard it and the number who have not. In a town of N people, suppose 1 person originates the rumor at time $t = 0$. Let $y = y(t)$ be the number of people who have heard the rumor at time t.*

   What does this model imply about the number of people who eventually have heard the rumor?

   Solution. The initial condition is $y(0) = 1$ since 1 person has heard the rumor at time 0. The rate of change is proportional to the product of the number of people who have heard it (y) and the number who have not ($N - y$), so $\boxed{\frac{dy}{dt} = ky(N - y)}$, where k is a positive constant.

   As long as not everybody has heard the rumor, the spreading rate is positive. Therefore, it seems like eventually everybody will have heard the rumor.

The following problems are about solutions to differential equations.

9. *Which of the following is a solution to $\frac{dy}{dx} = y$?*

   (a) $y = \frac{x^2}{2} + C$.

   (b) $y = e^x + C$.

   (c) $y = Ce^x$.

   Solution. $\boxed{\text{(c)}}$. If $y = Ce^x$, then $\frac{dy}{dx} = Ce^x = y$.

   If $y = \frac{x^2}{2} + C$, then $\frac{dy}{dx} = x$, which is not equal to y.

   If $y = e^x + C$, then $\frac{dy}{dx} = e^x$, which is not equal to y (unless $C = 0$).

10. *Give two solutions to $\frac{dy}{dx} = 5y$. What is the general solution?*

    Solution. A general solution is $\boxed{Ce^{5x}}$. Two specific solutions are e^{5x} and $-e^{5x}$.

11. *Give two solutions to to $\frac{dy}{dx} = 5x$. What is the general solution?*

    Solution. We actually know how to solve this already: the equation says we are looking for a function of x whose derivative is $5x$. That is, we want antiderivatives of $5x$, and we know that the general antiderivative is $\boxed{\frac{5}{2}x^2 + C}$. Two specific solutions are $\frac{5}{2}x^2$ and $\frac{5}{2}x^2 - 1$.

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[1] In reality, you cannot withdraw money continuously from the bank, but it's convenient to use a continuous model.

Slope Fields

1. Draw the slope fields for the following differential equations:

   (a) $\frac{dy}{dt} = 1$.

   (b) $\frac{dy}{dt} = t$.

   (c) $\frac{dy}{dt} = y$.

   (d) $\frac{dy}{dt} = \frac{-t}{y}$.

2. Draw the slope field for the differential equation $\frac{dy}{dt} = y - 1$. Sketch two solutions to the equation.

3. Which of the following is a solution to $\frac{dy}{dt} = y - 1$?

   (a) $y = Ce^t$

   (b) $y = Ce^t - t$

   (c) $y = Ce^{-t} - 1$

   (d) $y = Ce^t - 1$

   (e) $y = Ce^t + 1$

4. Which of the following is a solution to $y'' - y' - 6y = 0$?

   (a) $y = Ce^t$.

   (b) $y = C \sin 2t$.

   (c) $y = 5e^{3t} + e^{-2t}$.

   (d) $y = e^{3t} - 2$.

1. *Find the general solution of the differential equation* $\frac{dM}{dt} = 2.4 - .2M$. *(Such a differential equation came up, for instance, when we modeled the amount of medicine in a patient's body.)*

   Solution. We can use separation of variables: $\frac{1}{2.4-0.2M}\ dM = dt$.[1] Simplifying,

   $$-\frac{1}{0.2} \cdot \frac{1}{M-12}\ dM = dt.$$

   Integrating both sides,

   $$-\frac{1}{0.2} \ln|M-12| = t + C.$$

   Multiplying both sides by -0.2 gives

   $$\ln|M-12| = -0.2t - 0.2C.$$

   Since $-0.2C$ is just an arbitray constant, we can give it a new name; let's call it A. So,

   $$\ln|M-12| = A - 0.02t.$$

   Then,

   $$M - 12 = \pm e^A e^{-0.02t}.$$

   Again, $\pm e^A$ is just an arbitrary constant, so let's call it B. So,

   $$M - 12 = Be^{-0.02t},$$

   and $\boxed{M(t) = 12 + Be^{-0.02t}}$.[2]

2. *Last time, we solved the differential equation* $\frac{dy}{dt} = -\frac{t}{y}$ *by drawing the slope field, guessing the solution, and checking it. Now, solve the differential equation using separation of variables.*

   Solution. We can rewrite $\frac{dy}{dt} = -\frac{t}{y}$ as

   $$y\ dy = -t\ dt.$$

   Integrating both sides,

   $$\frac{1}{2}y^2 = -\frac{1}{2}t^2 + C.$$

   Multiplying both sides by 2,

   $$y^2 = -t^2 + 2C.$$

   Since $2C$ is still just an arbitrary constant, we can give it a new name; let's call it A. So,

   $$y^2 = A - t^2,$$

   and $\boxed{y = \pm\sqrt{A - t^2}}$.

[1] Technically, we can only do this if $2.4 - 0.2M \neq 0$; if $2.4 - 0.2M = 0$, which happens when $M = 12$, the original differential equation is just $\frac{dM}{dt} = 0$, so $M(t) = 12$ is a solution.

[2] Since $B = \pm e^A$, B should technically be non-zero. But we remarked earlier that $M(t) = 12$ is a solution, so $B = 0$ is also okay.

3. *Solve the differential equation* $\frac{dy}{dt} = e^{-t-y}$, *and find the particular solution satisfying the initial condition* $y(0) = 1$.

Solution. We can rewrite $\frac{dy}{dt} = e^{-t-y}$ as

$$e^y \, dy = e^{-t} \, dt.$$

Integrating both sides,

$$e^y = -e^{-t} + C.$$

So, $y(t) = \ln(C - e^{-t})$. Plugging in the initial condition gives $1 = \ln(C - 1)$, so $e = C - 1$, and $C = 1 + e$. So, our answer is $\boxed{y(t) = \ln(1 + e - e^{-t})}$.

4. *Solve the differential equation* $y' = 2y - 6$.

Solution. First, $y(t) = 3$ is a solution.

If $y \neq 3$, we can rewrite the differential equation as $\frac{dy}{dt} = 2y - 6$, or $\frac{1}{2y-6} \, dy = dt$. To integrate both sides, it's helpful to rewrite the left side as $\frac{1}{2} \cdot \frac{1}{y-3}$. So, we have

$$\begin{aligned}
\int \frac{1}{2} \frac{1}{y-3} \, dy &= \int \, dt \\
\frac{1}{2} \ln|y-3| &= t + C \\
\ln|y-3| &= 2t + A \text{ where } A = 2C \\
|y-3| &= e^A e^{2t} \\
y - 3 &= \pm e^A e^{2t} \\
y - 3 &= Be^{2t} \text{ where } B = \pm e^A \\
y &= \boxed{3 + Be^{2t}}
\end{aligned}$$

Here, B can be 0 (because we started out saying that $y(t) = 3$ is a solution), or it can be $\pm e^A$, which accounts for any positive or negative constant. So, B can be any constant.

5. *Which of the following differential equations are separable? (You need not solve.)*

   (a) $\frac{dy}{dt} = t + y$.

   (b) $\frac{dy}{dt} = \frac{y}{\sin t}$.

   (c) $\frac{dy}{dt} = \frac{\sin t}{y} + t$.

Solution. (a) is not separable.

(b) is separable, for we can rewrite it as $\frac{1}{y} \, dy = \frac{1}{\sin t} \, dt$.

(c) is not separable.

6. *A* 20*-quart juice dispenser in a cafeteria is filled with a juice mixture that is* 10% *mango and* 90% *orange juice. A pineapple-mango blend (*40% *pineapple and* 60% *mango) is entering the dispenser at a rate of* 4 *quarts an hour and the well-stirred mixture leaves at a rate of* 4 *quarts an hour. Model the situation with a differential equation whose solution,* $M(t)$, *is the amount of mango juice in the container at time* t. *(*$t = 0$ *is the time when the pineapple-mango blend starts to enter the dispenser.)*

Solution. Since $M(t)$ is the amount of mango juice in the container at time t, $\frac{dM}{dt}$ is the rate of change of the amount of mango juice in the container. We know that this is equal to (rate at which mango juice is entering the container) minus (rate at which mango juice is leaving the container).

Let's first focus on the stuff entering the container. This is a pineapple-mango blend, entering at a rate of 4 quarts per hour. Only 60% of this is mango juice though, so mango juice is entering the container at a rate of $.6 \cdot 4 = 2.4$ quarts per hour.

Now, let's look at the stuff exiting the container. This is a blend of all of the juices, and it's leaving at a rate of 4 quarts per hour. What we need to know is what percent of this blend is mango juice. This is simple if we think about what our variables mean: $M(t)$ is the amount of mango juice in the container at time t, while there is always 20 quarts of juice in all. So, the percentage of the 20 quarts which is mango juice is $\frac{M(t)}{20}$. Therefore, mango juice is leaving the container at a rate of $\frac{M(t)}{20} \cdot 4 = \frac{M(t)}{5}$ quarts per hour.

So, our final differential equation is $\boxed{\frac{dM}{dt} = 2.4 - \frac{M}{5}}$. We also have an initial condition, since we know how much mango juice is in the dispenser at the beginning: 10% of the 20 quarts, or 2 quarts. So, our initial condition is $M(0) = 2$.

7. *Suppose that, in the previous problem, the mixture was leaving at a rate of 5 quarts per hour rather than 4 quarts per hour. Model the new situation.*

Solution. We still need to use $\frac{dM}{dt}$ = (rate at which mango juice is entering the container) minus (rate at which mango juice is leaving the container), and the rate at which mango juice is entering the container is just like in the previous problem, 2.4 quarts per hour.

Let's look at the stuff exiting the container. Again, we need to use the formula (rate at which mango juice is leaving) = (concentration of mango juice in the mixture) times (rate at which mixture is leaving). The rate at which the mixture is leaving is 5 quarts per hour. The concentration of mango juice in the mixture is equal to (amount of mango juice in mixture) divided by (total amount of mixture). The amount of mango juice is exactly $M(t)$. The total amount of mixture is a little more complicated. At time $t = 0$, there is 20 quarts of juice in the container. However, because the juice is entering at a rate of 4 quarts per minute and leaving at 5 quarts per minute, there is a net loss of 1 quart per minute. Thus, after t minutes, the amount of juice in the container is $20 - t$. So, the concentration of mango juice in the mixture at time t is $\frac{M(t)}{20-t}$, and the rate at which mango juice is leaving the container is $\frac{M(t)}{20-t} \cdot 5$.

So, our final differential equation is $\boxed{\frac{dM}{dt} = 2.4 - \frac{5M}{20-t}}$. Again, we have the initial condition $M(0) = 2$.

Second-Order Homogeneous Differential Equations with Constant Coefficients

1. Suppose $f(t)$ and $g(t)$ are both solutions to the differential equation $y''+by'+cy=0$. Is $C_1 f(t)+C_2 g(t)$ a solution as well?

2. Can you guess solutions of $y''=y$? Try to guess two solutions that are not just multiples of each other.

3. Can you guess solutions of $y''=4y$? Try to guess two solutions that are not just multiples of each other.

4. Solve $y''-y'=6y$.

5. Solve $y'' + 5y' + 4y = 0$ where $y(0) = 1$ and $y'(0) = 2$.

6. Solve $y'' - 4y' + 4y = 0$.

7. Show that, if the characteristic equation $y'' + by' + cy = 0$ has one repeated root r, then $y = te^{rt}$ is a solution to $y'' + by' + cy = 0$.

1. *Solve* $y'' - 6y' + 9y = 0$.

   Solution. The characteristic equation is $r^2 - 6r + 9 = 0$, or $(r-3)^2 = 0$. Since $r = 3$ is a repeated root of this equation, the general solution is $\boxed{C_1 e^{3t} + C_2 t e^{3t}}$.

2. *Solve* $y'' + y = 0$.

   Solution. The characteristic equation is $r^2 + 1 = 0$, and the roots are $r = \pm i$. Since $e^{it} = \cos t + i \sin t$ and $e^{-it} = \cos(-t) + i\sin(-t) = \cos t - i \sin t$, two different solutions are $\cos t$ and $\sin t$. Thus, the general solution is $\boxed{y(t) = C_1 \cos t + C_2 \sin t}$.

3. *Solve* $y'' - 2y' + 5y = 0$.

   Solution. The characteristic equation is $r^2 - 2r + 5 = 0$. The roots of this are

$$\begin{aligned}\frac{2 \pm \sqrt{2^2 - 4(1)(5)}}{2} &= \frac{2 \pm \sqrt{-16}}{2} \\ &= \frac{2 \pm 4i}{2} \\ &= 1 \pm 2i\end{aligned}$$

   So, we know two solutions are $e^{(1+2i)t}$ and $e^{(1-2i)t}$, and the general solution is $C_1 e^{(1+2i)t} + C_2 e^{(1-2i)t}$. We're only interested in the real solutions, so let's rewrite $e^{(1+2i)t}$ and $e^{(1-2i)t}$ to find the real solutions:

$$\begin{aligned}e^{(1+2i)t} &= e^{t+2it} \\ &= e^t \cdot e^{i(2t)} \\ &= e^t(\cos 2t + i \sin 2t) \\ &= e^t \cos 2t + i e^t \sin 2t\end{aligned}$$

   Similarly, $e^{(1-2i)t} = e^t \cos 2t - i e^t \sin 2t$.

   Our general solution was $C_1 e^{(1+2i)t} + C_2 e^{(1-2i)t}$, and we now know that we can rewrite this as $C_1 \left(e^t \cos 2t + i e^t \sin 2t\right) + C_2 \left(e^t \cos 2t - i e^t \sin 2t\right)$. Re-grouping the terms, we see that we can write this as $\boxed{A_1 e^t \cos 2t + A_2 e^t \sin 2t}$.

4. (a) *Solve* $y'' + 2y' + 4y = 0$ *with initial conditions* $y(0) = 1$ *and* $y'(0) = 0$.

   Solution. The characteristic equation $r^2 + 2r + 4 = 0$ has roots $r = \frac{-2 \pm \sqrt{4-16}}{2} = -1 \pm \sqrt{3}i$. Therefore, two different solutions are $e^{(-1+\sqrt{3}i)t}$ and $e^{(-1-\sqrt{3}i)t}$. We rewrite:

$$\begin{aligned}e^{(-1+\sqrt{3}i)t} &= e^{-t+\sqrt{3}ti} \\ &= e^{-t} \cdot e^{i(\sqrt{3}t)} \\ &= e^{-t}\left[\cos\left(\sqrt{3}t\right) + i \sin\left(\sqrt{3}t\right)\right]\end{aligned}$$

   Following the same reasoning as in the previous problem, we see that the general solution is $y(t) = C_1 e^{-t} \cos\left(\sqrt{3}t\right) + C_2 e^{-t} \sin\left(\sqrt{3}t\right)$.

   Let's use the initial conditions to solve for C_1 and C_2. The initial condition $y(0) = 1$ tells us $C_1 \cos(0) + C_2 \sin(0) = 1$. Since $\cos(0) = 1$ and $\sin(0) = 0$, we know that $C_1 = 1$.

To use the second initial condition, we first need to differentiate $y(t)$:

$$y'(t) = C_1\left[-e^{-t}\cos\left(\sqrt{3}t\right) - \sqrt{3}e^{-t}\sin\left(\sqrt{3}t\right)\right] + C_2\left[-e^{-t}\sin\left(\sqrt{3}t\right) + \sqrt{3}e^{-t}\cos\left(\sqrt{3}t\right)\right].$$

Therefore, $y'(0) = -C_1 + \sqrt{3}C_2$. So, $0 = -1 + \sqrt{3}C_2$, and $C_2 = \frac{1}{\sqrt{3}}$.

Thus, our final solution is $\boxed{y(t) = e^{-t}\cos\left(\sqrt{3}t\right) + \frac{1}{\sqrt{3}}e^{-t}\sin\left(\sqrt{3}t\right)}$.

(b) *Interpret part (a) in terms of a vibrating spring. What is happening to the spring as time goes on?*

Solution. The differential equation $y'' + 2y' + 4y = 0$ describes a vibrating spring with friction. The initial condition $y(0) = 1$ says that the spring is initially stretched 1 unit beyond its equilibrium position, and the initial condition $y'(0) = 0$ says that its initial velocity is 0.

Our solution was $y(t) = e^{-t}\cos\left(\sqrt{3}t\right) + \frac{1}{\sqrt{3}}e^{-t}\sin\left(\sqrt{3}t\right)$, or $y(t) = e^{-t}\left[\cos\left(\sqrt{3}t\right) + \frac{1}{\sqrt{3}}\sin\left(\sqrt{3}t\right)\right]$. This function oscillates while decreasing in magnitude, and $\lim_{t\to\infty} y(t) = 0$.

5. *Which of the following differential equations has periodic solutions? What is the period?*

(a) $y'' + 2y' - 3y = 0$.

(b) $y'' + 2y + 3y = 0$.

(c) $y'' + 4y' = 0$.

(d) $y'' + 4y = 0$.

(e) $y'' - 4y = 0$.

Does this agree with your interpretation of the differential equations in terms of vibrating springs?

Solution. Let's look at each one.

(a) The characteristic equation is $r^2 + 2r - 3 = 0$, or $(r+3)(r-1) = 0$. So, the general solution is $C_1e^{-3t} + C_2e^t$, which is not periodic. This differential equation can't be interpreted in terms of vibrating springs ($y'' + by' + cy = 0$ is only the differential equation for a vibrating spring if $b \geq 0$ and $c > 0$).

(b) The characteristic equation is $r^2 + 2r + 3 = 0$, which has roots $r = \frac{-2\pm\sqrt{-8}}{2} = -1 \pm \sqrt{2}i$. Since $e^{(-1+\sqrt{2}i)t} = e^{-t}\cdot e^{i(\sqrt{2}t)} = e^{-t}(\cos\sqrt{2}t + i\sin\sqrt{2}t)$, if we follow the reasoning we used in #3, we find that the general solution is $C_1e^{-t}\cos(\sqrt{2}t) + C_2e^{-t}\sin(\sqrt{2}t)$. This is not periodic. This differential equation describes a vibrating spring subject to friction (the fact that $b > 0$ means there is friction), so it makes sense that the solution is not periodic. (Such a spring should vibrate less and less over time rather than vibrating the same amount forever.)

(c) The characteristic equation is $r^2 + 4r = 0$, or $r(r+4) = 0$. This has roots $r = 0$ and $r = -4$, so the general solution is $C_1 + C_2e^{-4t}$, which is not periodic. This differential equation can't be interpreted in terms of vibrating springs.

(d) The characteristic equation is $r^2 + 4 = 0$, or $r = \pm 2i$. Since $e^{2it} = \cos 2t + i\sin 2t$, if we follow the reasoning we used in #3, we find that the general solution is $C_1\cos(2t) + C_2\sin(2t)$. This is periodic with period π. This differential equation describes a vibrating spring without friction (since $b = 0$), and it makes sense that such a spring should oscillate back and forth periodically.

(e) The characteristic equation is $r^2 - 4 = 0$, or $(r-2)(r+2) = 0$. This has roots $r = 2$ and $r = -2$, so the general solution is $C_1 e^{-2t} + C_2 e^{-2t}$, which is not periodic. This differential equation can't be interpreted in terms of vibrating springs.

Second-Order Homogeneous Differential Equations with Constant Coefficients

1. Which of the following differential equations has periodic solutions? What is the period?

   (a) $y'' + 2y' - 3y = 0$.

   (b) $y'' + 2y' + 3y = 0$.

   (c) $y'' + 4y' = 0$.

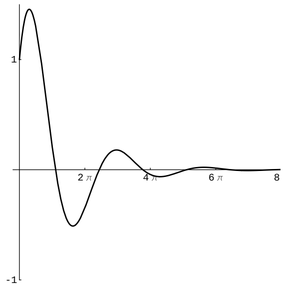
   (d) $y'' + 4y = 0$.

   (e) $y'' - 4y = 0$.

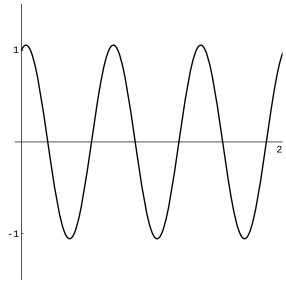
   Does this agree with your interpretation of the differential equations in terms of vibrating springs?

2. A spring with a mass of 5 kg has a natural length of 6 cm. A 20 N force is required to compress it to a length of 5 cm. If the spring is stretched to a length of 7 cm and released, find the position of the mass at time t. Sketch a graph of the position vs. time.

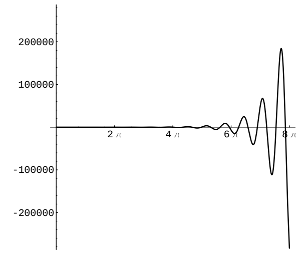
3. Match each differential equation with the graph of its solution. In each case, the differential equation has initial conditions $y(0) = 1$, $y'(0) = 1$.



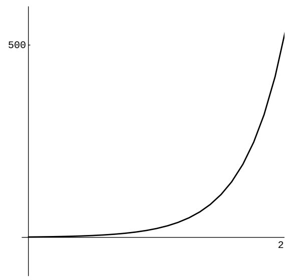
(1)



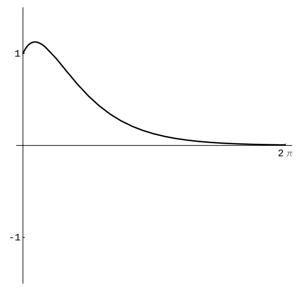
(2)



(3)



(4)



(5)

(a) $y'' + 3y' + 2y = 0.$

(b) $y'' + 9y = 0.$

(c) $y'' - 2y' + y = 0.$

(d) $y'' - y' + 10y = 0.$

(e) $y'' + \frac{1}{2}y' + \frac{5}{8}y = 0.$

4. Solve the differential equation $y'' + y = 0$ with initial conditions $y(0) = 0$ and $y'(0) = 1$.

Systems of Differential Equations

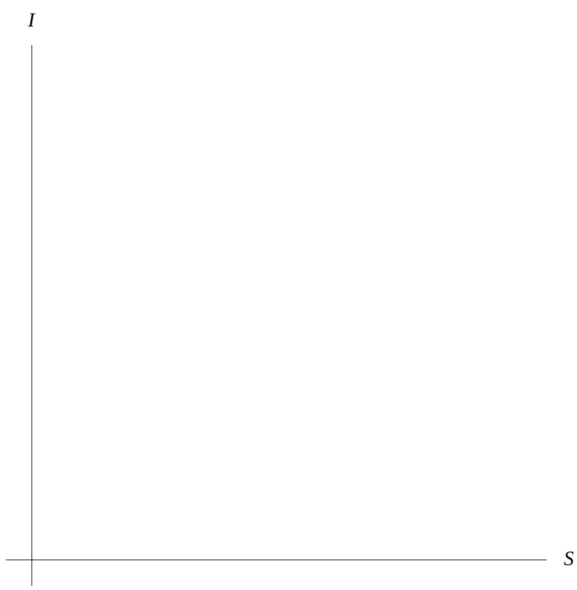
We've used systems of differential equations to model interaction between species. Systems can also be used to model disease epidemics.

Suppose that there is a large population of people, and some of the people have a fatal disease. This disease is infectious, so anybody who doesn't have the disease is susceptible to getting it. Let $I(t)$ be the number of people infected at time t, and let $S(t)$ be the number of people who are susceptible at time t.

1. How could you model this situation with a system of differential equations? You may ignore birth and death, except for death due to the disease, which you should include. (There are many many different answers; when in doubt, opt for simplicity.)

2. Using common sense, find the equilibrium points in this model. (You do not need to use the differential equations you found in #1; just think about the situation.)

3. Using common sense, sketch some typical phase trajectories in the phase plane.



4. A reasonable system for the situation described is:

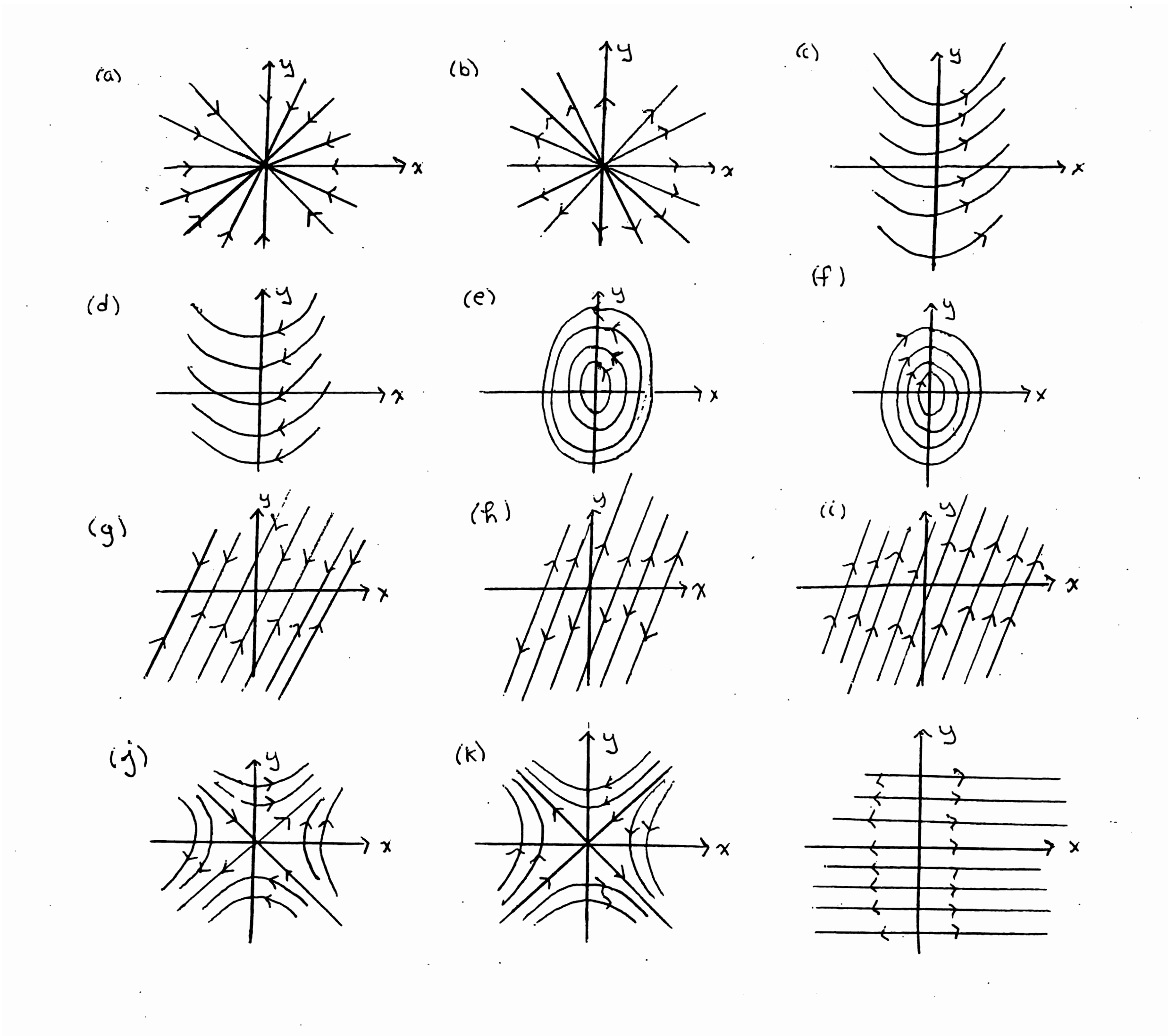
$$\begin{aligned}\frac{dS}{dt} &= -0.001IS \\ \frac{dI}{dt} &= 0.001IS - 0.1I\end{aligned}$$

Sketch the phase portrait for this system. (Be sure to draw the nullclines and equilibrium points.)

5. If the population starts with 50 infected people and 200 susceptible people, what will happen in the long run?

Systems of Differential Equations

The problems refer to these diagrams:

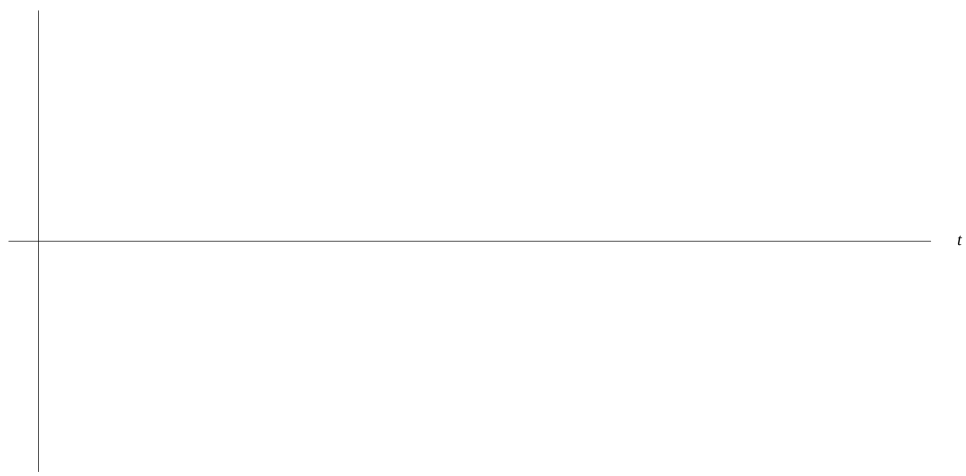


1. Find the diagram which matches the system.

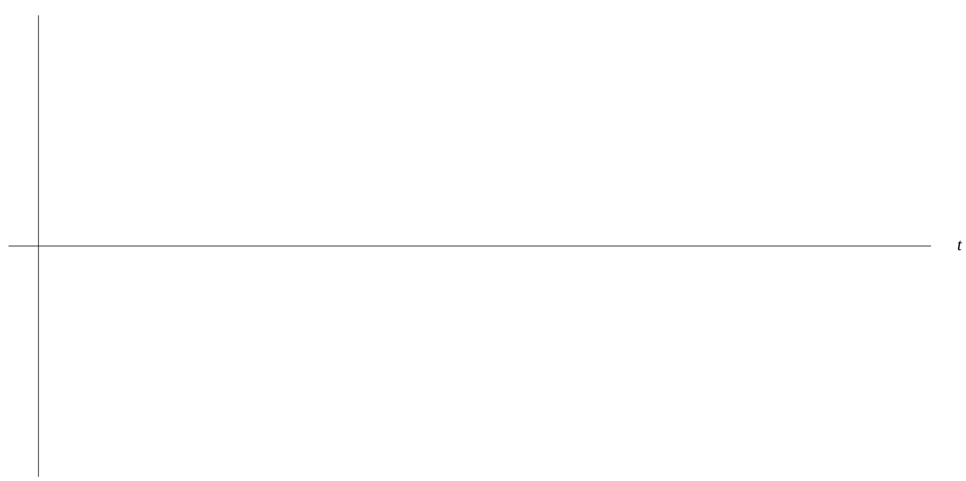
   (i) $\frac{dx}{dt} = y$ and $\frac{dy}{dt} = -2x$.

   (ii) $\frac{dx}{dt} = y$ and $\frac{dy}{dt} = 3y$.

2. (i) Suppose that the system $\frac{dx}{dt} = f(x, y)$ and $\frac{dy}{dt} = g(x, y)$ has phase portrait (e). Sketch possible graphs of $x(t)$ and $y(t)$, assuming $x(0) = 0$ and $y(0) = 3$.



   (ii) Do the same thing for (g).



3. Solve the system $\frac{dx}{dt} = y$ and $\frac{dy}{dt} = 3y$ with initial conditions $x(0) = 0$ and $y(0) = 3$.

4. Solve the system $\frac{dx}{dt} = y$ and $\frac{dy}{dt} = -2x$ with initial conditions $x(0) = 0$ and $y(0) = 3$.

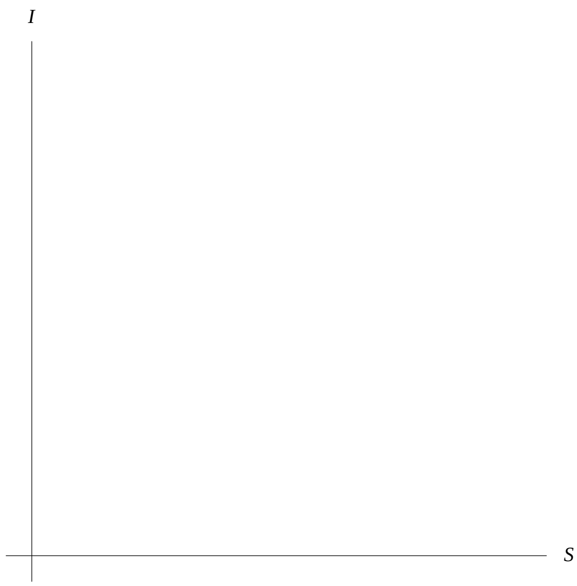
5. Find a system of differential equations whose phase portrait looks like (l), the last diagram.

Measles

We'll look at the system

$$\frac{dS}{dt} = -IS + 50$$
$$\frac{dI}{dt} = IS - 10I$$

1. Do a qualitative phase plane analysis of this system. (You should draw equilibrium points, nullclines, and the direction of the trajectories in each region.)



2. Based on your phase plane analysis, what do you think the trajectories look like? Sketch a possible trajectory on your diagram if $S(0) = 5$ and $I(0) = 20$.

3. Using your trajectory, sketch a possible graph of $I(t)$ if $S(0) = 5$ and $I(0) = 20$.

4. This graph shows the number of cases of measles in 2 week periods in London from 1944 to 1966. Does our system give the same qualitative behavior?