Last Homework Problem of the Term

1. A vibrating spring can be modeled by a differential equation of the form $x'' + bx' + cx = 0$ where b and c are non-negative constants. The case $b = 0$ corresponds to a frictionless system. In this problem you'll consider the differential equations below in a couple of different ways, making a connection between the second order equations and analyzing systems of differential equations.

$$\text{(i)}\ x'' + \tfrac{1}{4}x = 0 \qquad \text{(ii)}\ x'' + \tfrac{1}{10}x' + \tfrac{1}{4}x = 0$$

We'll begin with equation (i), corresponding to a frictionless system

(a) Solve the second order differential equation.
Find the particular solution corresponding to the initial conditions $x(0) = 1$ and $x'(0) = 0$.
Sketch the solution.

(b) Equation (i) can be expressed $\frac{d^2x}{dt^2} + \frac{1}{4}x = 0$. Since $x(t)$ gives position at time t, $\frac{dx}{dt}$ gives velocity. Let $v = \frac{dx}{dt}$ (*).
Then the differential equation can be written

$$\frac{dv}{dt} + \frac{1}{4}x = 0 \qquad (**)$$

Using both (*) and (**) gives the system of first order differential equations:

$$\frac{dx}{dt} = v \qquad \frac{dv}{dt} = -\frac{1}{4}x.$$

Analyze this system in the phase plane. Check your work using the pplane applet.

(c) Using the system

$$\frac{dx}{dt} = v \qquad \frac{dv}{dt} = -\frac{1}{4}x,$$

and the relationship $\dfrac{dv}{dx} = \dfrac{\frac{dv}{dt}}{\frac{dx}{dt}}$, express $\frac{dv}{dx}$ as a function of x and v. The resulting differential equation is separable. Solve.
Conclude that the trajectories in the phase plane are closed curves. Does this make sense given a frictionless model?

(d) Which trajectory in the phase-plane corresponds to the solution you drew in part (a)?

Now turn to equation (ii) $x'' + \frac{1}{10}x' + \frac{1}{4}x = 0$.

(e) Solve $x'' + \frac{1}{10}x' + \frac{1}{4}x = 0$.
Find the particular solution corresponding to the initial conditions $x(0) = 1$ and $x'(0) = 0$.
Sketch the solution.

(f) Since $x(t)$ gives position at time t, $\frac{dx}{dt}$ gives velocity. Let $v = \frac{dx}{dt}$ (*).
Then the differential equation $\frac{d^2x}{dt^2} + \frac{1}{10}\frac{dx}{dt} + \frac{1}{4}x = 0$ can be written

$$\frac{dv}{dt} + \frac{1}{10}v + \frac{1}{4}x = 0 \qquad (**)$$

Using both (*) and (**) gives the system of first order differential equations:

$$\frac{dx}{dt} = v \qquad \frac{dv}{dt} = -\frac{1}{10}v - \frac{1}{4}x.$$

Analyze this system in the phase plane. Check your work using the pplane applet.

(g) Using the system

$$\frac{dx}{dt} = v \qquad \frac{dv}{dt} = -\frac{1}{10}v - \frac{1}{4}x,$$

and the relationship $\frac{dv}{dx} = \frac{\frac{dv}{dt}}{\frac{dx}{dt}}$, express $\frac{dv}{dx}$ as a function of x and v. Is the resulting differential equation separable? Solve only if it is separable.

(h) Use the pplane applet to sketch a trajectory in the phase-plane corresponding to the solution you drew in part (f). Do your pictures makes physical sense?