1 General Series: Convergence and Divergence

An infinite series, $\sum_{k=1}^{\infty} a_k$, converges if its sequence of partial sums converges to a finite number. In other words, to determine the convergence of a series, look at $\lim_{n\to\infty} s_n$ where $s_n = \sum_{k=1}^{n} a_k$.

We know that if the sequence of partial sums is increasing and bounded the series converges; if the sequence of partial sums is increasing and unbounded then the series diverges. This is what comparison tests are based on (and why they apply only to series where the terms are eventually all positive).

1.1 Convergence Tests

Series that we know about:

- *Geometric Series:* A geometric series is a series of the form $\sum_{n=0}^{\infty} ar^n$. The series diverges if $|r| > 1$ and converges if $|r| < 1$. In the latter case the sum of the entire series is $\frac{a}{1-r}$ where a is the first term of the series and r is the common ratio.

  We proved this by writing the partial sums in closed form and computing a limit.

- *p-Series:* The series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$ and diverges otherwise.

  We proved this using the Integral Test.

Intrinsic Tests that can be used for all series without restiction

- *Nth Term Test for Divergence:* If $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ diverges.

  Note: If $\lim_{n\to\infty} a_n = 0$ we know nothing: the series can either converge or diverge. (Think about the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$)

- *Ratio Test:* Given $\sum_{n=1}^{\infty} a_n$, look at $\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}|$. If this ratio is less than 1, the series converges absolutely. If this ratio is greater than 1, the series diverges. If it equals 1, no conclusion can be drawn

  (Note that if a series converges conditionally then the Ratio Test will be inconclusive. Can you come up with examples to show this? The Ratio Test works well for series involving factorials and where the n is in an exponent. It can always be used when trying to determine a radius or interval of convergence).

Comparison Tests: Applicable only if terms are positive

- *Direct Comparison Test:* If a series $\sum_{n=1}^{\infty} a_n$ has all positive terms, and all of its terms are eventually bigger than those in a series that is known to be divergent, then it is also divergent. The reverse is also true–if all the terms are eventually smaller than those of some convergent series, then the series is convergent.

That is, if $\sum a_n$, $\sum b_n$ and $\sum c_n$ are all series with positive terms and $a_n \leq b_n \leq c_n$ for all n sufficiently large, then

if $\sum c_n$ converges, then $\sum b_n$ does as well
if $\sum a_n$ diverges, then $\sum b_n$ does as well.

(This is a good test to use with rational functions. Specifically, if the degree of the denominator is more than 1 greater than the degree of the numerator, try to prove that the series converges (compare with a p-series). In other cases, including when the difference in degree is exactly 1, prove that it diverges).

- *Limit Comparison Test:* Use this when you know what you want to compare to but the inequalites go the wrong way. Suppose $\sum a_n$ and $\sum b_n$ are series with positive terms. If $\lim_{n\to\infty} \frac{a_n}{b_n} = c$ where c is finite and non-zero, then either both the series converge or both the series diverge.

- *Integral Test:* Let $a_n = f(n)$. Then, if f is continuous, decreasing, and positive on $[1, \infty)$, we have that $\sum_{n=1}^{\infty} a_n$ and $\int_1^{\infty} f(x)\,dx$ either both converge or both diverge.

Alternating Series Test: Test (with error estimate)

- *Alternating Series Test:* If the terms in a series are (i) alternating in sign, (ii) approaching 0 and (iii) decreasing in absolute value, then the series converges. This is a test for convergence only. Don't use it and conclude a series diverges!

  If a series satifies the three conditions of the AST, then the error in using a partial sum to approximate the actual sum is bounded by the magnitude of the first unused term.

1.2 General information about convergence

- Partial Sums: Be able to distinguish between the terms of a series and the partial sums of the series. If the series converges we know the terms must tend towards zero whereas the partial sums must tend toward the sum of the series.

- Absolute Convergence:
  - Any series $\sum_{n=0}^{\infty} a_n$ either converges or diverges.
  - If $\sum_{n=0}^{\infty} a_n$ converges and $\sum_{n=0}^{\infty} |a_n|$ converges as well, we say the series $\sum_{n=0}^{\infty} a_n$ converges absolutely.
  - If $\sum_{n=0}^{\infty} a_n$ converges but $\sum_{n=0}^{\infty} |a_n|$ diverges, then we say $\sum_{n=0}^{\infty} a_n$ converges but it does not converge absolutely. (This is known as conditional convergence.)

- Multiplying by constants and leaving off the first few terms have *no effect* on whether a series converges. Also, adding a convergent series to another series will not change whether the other one converges.

1.3 Using Convergence Tests: A Strategy

Given a series, how do we know what to think about: p-series, geometric series, Nth Term Test for Divergence, Direct Comparison, Limit Comparison, Integral Test, Ratio Test, Alternating Series Test .... There is no definitive answer, but the following guidelines may be useful.

1. Is it a familiar series?

   A geometric series is a series of the form $\sum_{n=0}^{\infty} ar^n$; it converges if and only if $|r| < 1$.

   A $p-$Series is a series of the form $\sum_{n=1}^{\infty} \frac{1}{n^p}$; it converges if and only if $p > 1$.

2. Does $\lim_{n\to\infty} a_n \neq 0$? If so then by the Nth Term Test for Divergence the series diverges and you're done.

3. Does the series look similar to a geometric or a $p-$ series *and* are the terms of the series eventually all positive? Then use the Direct Comparison Test or the Limit Comparison Test.

   If the terms are not eventually all positive then you can look at $\sum |a_n|$ and test for absolute convergence. If the series converges absolutely, then it converges. Before doing this however, see if the series is alternating, and if so, try the alternating series test.

4. If the series is alternating? See if you can apply the Alternating Series Test.

5. Can we apply the ratio test? Series involving factorials or constants raised to a variable power readily lend themselves to be tested using the Ratio Test. (If the series is not initially composed of positive terms (or terms that eventually are positive) then the ratio test can be used to test for absolute convergence.) Keep in mind that the Ratio Test will not help us with $p-$series and therefore for series that are rational functions of n.

6. If all else fails (or at least doesn't work easily) and if $a_n = f(n)$ where f is continuous, decreasing, and positive on $[a, \infty)$, and we can evaluate $\int_a^{\infty} f(x)\, dx$, then the Integral Test can be applied.

Pointers:

- Do keep track of what you're doing and make your reasoning transparent by calling upon the test you are using.
- Think of problems about convergence as an exercise in logic. Don't assume information that you don't have.
- Don't assert that if the terms of a series go towards zero the series converges to zero or for that matter, that it converges at all. Have a counterexample handy. Similarly, don't assert that if the terms of a series go towards 1/2 that the series converges to 1/2. Understand why these are egregious errors!
- Don't use the comparison test to compare alternating series. (Remember that the comparison test rests on the notion of an increasing sequence of partial sums that is either bounded or unbounded.)

2 Power Series, Taylor Series

2.1 Power Series

A power series about $x = a$ is a series of the form $\sum_{n=0}^{\infty} c_n(x-a)^n$. Its interval of convergence (the values of x for which it converges) is one of the following:

- all x: $(-\infty, \infty)$ Radius of convergence : ∞
- only at $x = a$, at the center of the series.   Radius of convergence : 0
- an interval centered about a, with radius R:

  $(a-R, a+R)$ or $[a-R, a+R)$ or $[a-R, a+R]$ or $(a-R, a+R]$

To find the interval of convergence you can use the Ratio test. (If the power series is a geometric series, our results on geometric series can be used instead.) In the case that the radius of convergence is positive and finite the ratio test will not give information about the endpoints of this interval. You need to plug in those two endpoints separately and use some convergence test other than the Ratio Test to find the *interval of convergence*. (Why not use the Ratio Test? Well, it has just shown itself to be useless on the endpoints of the interval.)

2.2 Manipulating Power Series

Suppose that a power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ has radius of convergence R. Let

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \ldots + c_n(x-a)^n + \ldots = \sum_{n=0}^{\infty} c_n(x-a)^n$$

for $|x-a| < R$. Then

$$f'(x) = c_1 + 2c_2(x-a) + \ldots + nc_n(x-a)^{n-1} + \ldots = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1}$$

for $|x-a| < R$ and

$$\begin{aligned}
\int f(x)\,dx &= \int c_0\,dx + \int c_1(x-a)\,dx + \int c_2(x-a)^2\,dx + \ldots + \int c_n(x-a)^n\,dx + \ldots \\
&= C + c_0 x + c_1\frac{(x-a)^2}{2} + c_2\frac{(x-a)^3}{3} + \ldots + c_n\frac{(x-a)^{n+1}}{n+1} + \ldots \\
&= \sum_{n=0}^{\infty} c_n\frac{(x-a)^{n+1}}{n+1}
\end{aligned}$$

for $|x-a| < R$.

All of this is just exactly what you would hope for and should make you smile, if not jump for joy. Some of the delightful things about power series are that they are easy to evaluate, easy to integrate and easy to differentiate.

2.3 Taylor Series

The **nth degree Taylor polynomial generated by f about $x = a$** is the "best" nth degree polynomial approximation of f in the neighborhood of a. It is the polynomial that matches f at $x = a$ and whose first n derivatives match those of f at $x = a$.

(Why would you want to approximate a function by a polynomial? Because polynomials are tame and well-behaved. They are are easy to evaluate, easy to integrate and easy to differentiate.)

The nth degree Taylor polynomial generated by f about $x = a$ can be computed as follows:

$$\sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

The **Taylor series generated by f about $x = a$** is given by

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

Notice that a Taylor series is a power series, so everything we've said about power series holds for Taylor series.

How do you compute the Taylor series/ polynomial generated by f about $x = a$?

- You can compute it "from scratch." This means you need to take derivatives of f, evaluate them at $x = a$, establish a pattern, and then you're off and running.

  OR

- You can get a new Taylor series from a familiar one by substitution, multiplication, integration or differentiation, or a combination of these.

  (If you ever find yourself involved in a horrendous differentiation exercise, sit back and think about whether there is a clever way around this using one of these methods.)

Here are some Taylor expansions about $x = 0$ that you should know off the top of your head. They are part of your series vocabulary - they should be familiar and accessible:

- $\frac{1}{1-x} = 1 + x + x^2 + \ldots = \sum_{n=0}^{\infty} x^n$ for $-1 < x < 1$.
- $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$ for all x.
- $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$ for all x.
- $e^x = 1 + x + \frac{x^2}{2!} + \ldots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ for all x.
- $(1+x)^p = 1 + px + \frac{p(p-1)}{2!} + \frac{p(p-1)(p-2)}{3!} + \ldots = 1 + \sum_{n=1}^{\infty} \frac{(p)(p-1)\ldots(p-n+1)x^n}{n!}$

  for $-1 < x < 1$.

You can always get new Taylor series by adding, subtracting, multiplying by constants, differentiating, and integrating existing series. None of these operations will change the radius of convergence. You can also get new Taylor series by substituting into an old series, but this can change the radius of convergence. For example, the series for $\frac{1}{1-(2x)}$ is simply $\sum_{n=0}^{\infty}(2x)^n$, but it will only converge for $-1 < 2x < 1$, or $\frac{-1}{2} < x < \frac{1}{2}$.

2.4 Error

- When using a partial sum of a convergent *alternating series*, approximating error is relatively easy provided the terms are decreasing in magnitude. The size of the error is less than the first term that is not included in the approximation. When you are asked an error question, *always try to use this method if you can.*

- Given a Taylor polynomial approximation for which the AST error estimate cannot be applied you can use the Taylor Remainder Formula as follows:

$$f(x) = T_n(x) + R_n(x)$$

($R_n(x)$ denotes the error in approximation of $f(x)$ when you use up to the x^n term):

There exists a number c between x and a such that

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

Here, x is the value at which you are trying to approximate the function, a is the center of the Taylor series.

(Note that $R_n(x)$ sort of looks like the next term in the Taylor series, but differs in that it uses $f^{(n+1)}(c)$ instead of $f^{(n+1)}(a)$).

There are no instructions on how to find c, so the Taylor Remainder is used as follows: we bound $|R_n(x)|$. To do this find a constant M that is greater than or equal to $|f^{(n+1)}(c)|$ for all c between x and a. Then

$$0 \le |R_n(x)| = \frac{|f^{(n+1)}(c)|}{(n+1)!}|x-a|^{n+1} \le \frac{M}{(n+1)!}|x-a|^{n+1}.$$

We note that M is not unique. Find the best M you can without overexerting yourself!