7. $\dfrac{du}{dt} = 2 + 2u + t + tu \;\Rightarrow\; \dfrac{du}{dt} = (1+u)(2+t) \;\Rightarrow\; \displaystyle\int \dfrac{du}{1+u} = \int (2+t)dt \;\; [u \neq -1] \;\Rightarrow$

$\ln|1+u| = \tfrac{1}{2}t^2 + 2t + C \;\Rightarrow\; |1+u| = e^{t^2/2 + 2t + C} = Ke^{t^2/2 + 2t}$, where $K = e^C \;\Rightarrow\; 1 + u = \pm K e^{t^2/2 + 2t} \;\Rightarrow$

$u = -1 \pm Ke^{t^2/2 + 2t}$ where $K > 0$. $u = -1$ is also a solution, so $u = -1 + Ae^{t^2/2 + 2t}$, where A is an arbitrary constant.

12. $\dfrac{dP}{dt} = \sqrt{Pt} \;\Rightarrow\; dP/\sqrt{P} = \sqrt{t}\,dt \;\Rightarrow\; \int P^{-1/2}\,dP = \int t^{1/2}\,dt \;\Rightarrow\; 2P^{1/2} = \tfrac{2}{3}t^{3/2} + C.$

$P(1) = 2 \;\Rightarrow\; 2\sqrt{2} = \tfrac{2}{3} + C \;\Rightarrow\; C = 2\sqrt{2} - \tfrac{2}{3}$, so $2P^{1/2} = \tfrac{2}{3}t^{3/2} + 2\sqrt{2} - \tfrac{2}{3} \;\Rightarrow\; \sqrt{P} = \tfrac{1}{3}t^{3/2} + \sqrt{2} - \tfrac{1}{3} \;\Rightarrow$

$P = \left(\tfrac{1}{3}t^{3/2} + \sqrt{2} - \tfrac{1}{3}\right)^2.$

14. $\dfrac{dL}{dt} = kL^2 \ln t \;\Rightarrow\; \dfrac{dL}{L^2} = k \ln t\,dt \;\Rightarrow\; \displaystyle\int \dfrac{dL}{L^2} = \int k \ln t\,dt \;\Rightarrow\; -\dfrac{1}{L} = kt\ln t - \int k\,dt \;\;$ [by parts

with $u = \ln t,\ dv = k\,dt$] $\;\Rightarrow\; -\dfrac{1}{L} = kt\ln t - kt + C \;\Rightarrow\; L = \dfrac{1}{kt - kt\ln t - C}.$

$L(1) = -1 \;\Rightarrow\; -1 = \dfrac{1}{k - k\ln 1 - C} \;\Rightarrow\; C - k = 1 \;\Rightarrow\; C = k + 1.$ Thus, $L = \dfrac{1}{kt - kt\ln t - k - 1}.$

34. (a) Use 1 billion dollars as the x-unit and 1 day as the t-unit. Initially, there is \$10 billion of old currency in circulation, so all of the \$50 million returned to the banks is old. At time t, the amount of new currency is $x(t)$ billion dollars, so $10 - x(t)$ billion dollars of currency is old. The fraction of circulating money that is old is $[10 - x(t)]/10$, and the

amount of old currency being returned to the banks each day is $\dfrac{10 - x(t)}{10}0.05$ billion dollars. This amount of new

currency per day is introduced into circulation, so $\dfrac{dx}{dt} = \dfrac{10 - x}{10} \cdot 0.05 = 0.005(10 - x)$ billion dollars per day.

(b) $\dfrac{dx}{10 - x} = 0.005\,dt \;\Rightarrow\; \dfrac{-dx}{10 - x} = -0.005\,dt \;\Rightarrow\; \ln(10 - x) = -0.005t + c \;\Rightarrow\; 10 - x = Ce^{-0.005t}$, where

$C = e^c \;\Rightarrow\; x(t) = 10 - Ce^{-0.005t}.$ From $x(0) = 0$, we get $C = 10$, so $x(t) = 10\left(1 - e^{-0.005t}\right).$

(c) The new bills make up 90% of the circulating currency when $x(t) = 0.9 \cdot 10 = 9$ billion dollars.

$9 = 10\left(1 - e^{-0.005t}\right) \;\Rightarrow\; 0.9 = 1 - e^{-0.005t} \;\Rightarrow\; e^{-0.005t} = 0.1 \;\Rightarrow\; -0.005t = -\ln 10 \;\Rightarrow$

$t = 200\ln 10 \approx 460.517$ days ≈ 1.26 years.

38. (a) If $y(t)$ is the amount of salt (in kg) after t minutes, then $y(0) = 0$ and the total amount of liquid in the tank remains constant at 1000 L.

$$\frac{dy}{dt} = \left(0.05\,\frac{\text{kg}}{\text{L}}\right)\left(5\,\frac{\text{L}}{\text{min}}\right) + \left(0.04\,\frac{\text{kg}}{\text{L}}\right)\left(10\,\frac{\text{L}}{\text{min}}\right) - \left(\frac{y(t)}{1000}\,\frac{\text{kg}}{\text{L}}\right)\left(15\,\frac{\text{L}}{\text{min}}\right)$$

$$= 0.25 + 0.40 - 0.015y = 0.65 - 0.015y = \frac{130 - 3y}{200}\,\frac{\text{kg}}{\text{min}}$$

Hence, $\displaystyle\int \frac{dy}{130 - 3y} = \int \frac{dt}{200}$ and $-\frac{1}{3}\ln|130 - 3y| = \frac{1}{200}t + C$. Because $y(0) = 0$, we have $-\frac{1}{3}\ln 130 = C$,

so $-\frac{1}{3}\ln|130 - 3y| = \frac{1}{200}t - \frac{1}{3}\ln 130 \;\Rightarrow\; \ln|130 - 3y| = -\frac{3}{200}t + \ln 130 = \ln\left(130e^{-3t/200}\right)$, and

$|130 - 3y| = 130e^{-3t/200}$. Since y is continuous, $y(0) = 0$, and the right-hand side is never zero, we deduce that

$130 - 3y$ is always positive. Thus, $130 - 3y = 130e^{-3t/200}$ and $y = \frac{130}{3}\left(1 - e^{-3t/200}\right)$ kg.

(b) After one hour, $y = \frac{130}{3}\left(1 - e^{-3\cdot 60/200}\right) = \frac{130}{3}\left(1 - e^{-0.9}\right) \approx 25.7$ kg.

Note: As $t \to \infty$, $y(t) \to \frac{130}{3} = 43\frac{1}{3}$ kg.

14. (a) Let $T(t) =$ temperature after t minutes. Newton's Law of Cooling implies that $\dfrac{dT}{dt} = k(T - 5)$. Let $y(t) = T(t) - 5$.

Then $\dfrac{dy}{dt} = ky$, so $y(t) = y(0)e^{kt} = 15e^{kt} \;\Rightarrow\; T(t) = 5 + 15e^{kt} \;\Rightarrow\; T(1) = 5 + 15e^k = 12 \;\Rightarrow\; e^k = \frac{7}{15} \;\Rightarrow$

$k = \ln\frac{7}{15}$, so $T(t) = 5 + 15e^{\ln(7/15)t}$ and $T(2) = 5 + 15e^{2\ln(7/15)} \approx 8.3°$C.

(b) $5 + 15e^{\ln(7/15)t} = 6$ when $e^{\ln(7/15)t} = \frac{1}{15} \;\Rightarrow\; \ln\left(\frac{7}{15}\right)t = \ln\frac{1}{15} \;\Rightarrow\; t = \dfrac{\ln\frac{1}{15}}{\ln\frac{7}{15}} \approx 3.6$ min.

16. $\dfrac{dT}{dt} = k(T - 20)$. Let $y = T - 20$. Then $\dfrac{dy}{dt} = ky$, so $y(t) = y(0)e^{kt}$. $y(0) = T(0) - 20 = 95 - 20 = 75$,

so $y(t) = 75e^{kt}$. When $T(t) = 70$, $\dfrac{dT}{dt} = -1°$C/min. Equivalently, $\dfrac{dy}{dt} = -1$ when $y(t) = 50$. Thus,

$-1 = \dfrac{dy}{dt} = ky(t) = 50k$ and $50 = y(t) = 75e^{kt}$. The first relation implies $k = -1/50$, so the second relation says

$50 = 75e^{-t/50}$. Thus, $e^{-t/50} = \frac{2}{3} \;\Rightarrow\; -t/50 = \ln\left(\frac{2}{3}\right) \;\Rightarrow\; t = -50\ln\left(\frac{2}{3}\right) \approx 20.27$ min.

Differential Equations Handout A

8. (a) $\frac{dM}{dt} = kM$.

$$k = \frac{r}{M} = \frac{250}{5000} = \frac{1}{20}.$$

Thus the differential equation is

$$\frac{dM}{dt} = \frac{M}{20}.$$

(b) The general solution for $\frac{dB}{dt} = kB$ is $B = Ce^{kt}$.

$$B(0) = C = 600$$

and

$$B(10) = 600e^{10k} = 800 \Rightarrow k = \frac{\ln(\frac{800}{600})}{10} = \frac{1}{10}\ln\left(\frac{4}{3}\right).$$

So

$$B(t) = 600e^{\frac{1}{10}\ln\left(\frac{4}{3}\right)t}.$$

9. (a) $\frac{dP}{dt} = 0.03P - 6000$.

(b) $\frac{dP}{dt} = 0.03(P - 200,000)$

$$u = P - 200,000$$
$$\frac{du}{dt} = \frac{dP}{dt} = 0.03u$$
$$u = Ce^{0.03t}$$
$$P = Ce^{0.03t} + 200,000 \quad P(0) = C + 200,000 = 3,000,000 \Rightarrow C = 2,800,000$$
$$P = 2,800,000e^{0.03t} + 200,000$$

10. (a) (i) $y(0) = 0$ means the solution is the equilibrium line. It is a horizontal line at y=0.

(ii) $y(0) = 0.01$ means the solution is an exponential curve that is always positive and increasing.

(iii) $y(0) = -0.01$ means that the solution is an exponential curve that is always negative and increasing.
(b) unstable equilibrium (or semi-stable).

(c)
$$\frac{dy}{dx} = y^2 \to \frac{dy}{y^2} = 1dx \to \int \frac{1}{y^2}\,dy = \int dx$$
$$\to -\frac{1}{y} = x + C \to y = -\frac{1}{x + C} \to C = -1$$
$$y = -\frac{1}{x - 1}.$$

(d) As x goes to 1, y is an asymptote.

11. (a)
$$\frac{dW}{dt} = \left(0.2\frac{L\ white}{L\ total}\right)\left(2\frac{L\ total}{hour}\right) - \left(\frac{W}{10}\frac{L\ white}{L\ total}\right)\left(2\frac{L\ total}{hour}\right)$$
$$W(0) = 0.$$

(b)
$$\frac{dB}{dt} = \left(0.8\frac{L\ blue}{L\ total}\right)\left(2\frac{L\ total}{hour}\right) - \left(\frac{B}{10}\frac{L\ blue}{L\ total}\right)\left(2\frac{L\ total}{hour}\right)$$
$$B(0) = 10.$$

12. $f(x)$ cannot ever be both increasing and concave up because then both $y' > 0$ and $y'' > 0$. So then $y'' + y' > 0$, but $y'' + y' = -x^2 \le 0$.

Supplement 31.3

1(b).

$$\frac{dy}{dt} = y^2 - 4 = (y - 2)(y + 2).$$

The constant solutions (equilibriums) is given by setting the derivative $\frac{dy}{dt} = 0$, i.e. $(y - 2)(y + 2) = 0$, i.e. $y = 2$ and $y = -2$. These two lines decompose the $t - y$-plane into three parts: $y > 2$, $-2 < y < 2$ and $y < -2$. When $y > 2$, the slope $\frac{dy}{dt} > 0$. Thus if the initial condition $y(0) > 2$, the solution goes to infinity as $t \to \infty$. When $-2 < y < 2$, the slope $\frac{dy}{dx} < 0$. Thus if $-2 < y(0) < 2$, the solution $y \to -2$ as $t \to \infty$. Finally if $y < -2$, $\frac{dy}{dx} > 0$. Thus if $y(0) < -2$, the solution $y \to -2$ as $x \to \infty$. Therefore $y = 2$ is an unstable equilibrium and $y = -2$ is a stable equilibrium.

2. (a) We want $y = 3$ to be a constant solution (equilibrium), the slope y' to be negative if $y > 3$ and the slope y' to be positive if $y < 3$. One possible differential equation could be

$$\frac{dy}{dt} = 3 - y.$$

(b) By the same idea as in part (a), one can take

$$\frac{dy}{dt} = y - 3.$$

(c) $\frac{dy}{dt} = y(2 - y)$.

(d) $\frac{dy}{dt} = (y - 2)(y + 2) = y^2 - 4$. (Cf. **1(b)** above.)

3. (a) $y = 3$ is the only equilibrium. It is a stable one.

(b) $y = 3$ is the only equilibrium. It is an unstable one.

(c) $y = 2$ is the unique stable equilibrium; $y = 0$ is the unique unstable equilibrium.

(d) $y = 2$ is the unique unstable equilibrium; $y = -2$ is the unique stable equilibrium.

6. (a)(b)(c)(e) When $y = 0$, the slope $\frac{dy}{dx}$ of the solution curve $y(t)$ should be negative.

16. Omitted.

22. (a)

$$\begin{aligned}
\frac{dP(t)}{dt} &= 0.02 \cdot 10^3 - \frac{P(t)}{10^{10}} \cdot 10^3 \\
&= 20 - \frac{P(t)}{10^7} \\
&= 10^{-7} \left(2 \cdot 10^8 - P(t) \right).
\end{aligned}$$

(b) Set the right hand side of the above differential equation $10^{-7} \left(2 \cdot 10^8 - P(t) \right) = 0$, and then we see the only constant solution is $P(t) = 2 \cdot 18^8$.

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(d) In this case,

$$\text{rate in} = 0.02 \cdot 10^2$$
$$= 2.$$
$$\text{rate out} = \frac{P(t)}{10^{10} + 10^2 t - 10^3 t} \cdot 10^3$$
$$= \frac{10^3 P(t)}{10^{10} - 900t}.$$

Thus the new differential equation is

$$\frac{dP(t)}{dt} = 2 - \frac{10^3 P(t)}{10^{10} - 900t}.$$