2. (a) $dx/dt = 0.12x - 0.0006x^2 + 0.00001xy$. $dy/dt = 0.08y + 0.00004xy$.

The xy terms represent encounters between the two species x and y. An increase in y makes dx/dt (the growth rate of x) larger due to the positive term $0.00001xy$. An increase in x makes dy/dt (the growth rate of y) larger due to the positive term $0.00004xy$. Hence, the system describes a cooperation model.

(b) $dx/dt = 0.15x - 0.0002x^2 - 0.0006xy = 0.15x(1 - x/750) - 0.0006xy$.

$dy/dt = 0.2y - 0.00008y^2 - 0.0002xy = 0.2y(1 - y/2500) - 0.0002xy$.

The system shows that x and y have carrying capacities of 750 and 2500. An increase in x reduces the growth rate of y due to the negative term $-0.0002xy$. An increase in y reduces the growth rate of x due to the negative term $-0.0006xy$. Hence, the system describes a competition model.

Differential Equations Handout A

13. (a) The nullclines are given by $x = 0$ and $y = 0$ and the equilibrium is then the origin $(0, 0)$. They decompose the xy plane into the standard four quadrants. On the nullcline $x = 0$, $\frac{dy}{dt} = 0$, so we have horizontal tangents there. When $y > 0$, $\frac{dx}{dt} < 0$ and when $y < 0$, $\frac{dx}{dt} > 0$.

Similarly on the nullcline $y = 0$, $\frac{dx}{dt} = 0$ so we have vertical tangents there. For $x > 0$, $\frac{dy}{dt} > 0$, and for $x < 0$, $\frac{dx}{dt} < 0$.

In the first quadrant $x > 0, y > 0$, $\frac{dx}{dt} < 0$, $\frac{dy}{dt} > 0$ so the general arrow goes upper-left. Similarly in the second quadrant, the arrow goes lower-left; in the third quadrant, it goes lower-right; in the forth quadrant, it goes upper-right. See a picture in p.4.

(b) For $x = 4\cos t$ and $y = 4\sin t$,

$$\begin{aligned} \frac{dx}{dt} &= -4\sin t = -y, \\ \frac{dy}{dt} &= 4\cos t = x. \end{aligned}$$

Also

$$x^2 + y^2 = 16\cos^2 t + 16\sin^2 t = 16.$$

14. (a) Competitive because the xy terms have negative coefficients in both cases.

(b) The equilibriums are the common solutions of

$$\begin{aligned} 0 &= 0.1x - 0.05xy = 0.05x(2 - y) \\ 0 &= 0.1y - 0.05xy = 0.05y(2 - x), \end{aligned}$$

which are either $(0, 0)$ or $(2, 2)$.

(c) The nullclines are either $0.05x(2 - y) = 0$, i.e. $x = 0$ or $y = 2$, or $0.05y(2 - x) = 0$, i.e. $y = 0$ or $x = 2$.

Now let us restrict to the first quadrant since this is a model of poputations. In the case that $0.05x(2-y) = 0$, we have vertical tangents since the derivative of x is 0. On the line $x = 0$ in the first quadrant, the flow goes upwards since in this case $y' = 0.1y > 0$. Similarly when $y = 2$, the flow goes upwards if $x < 2$ and downwards if $x > 2$.

On the other hand, if $0.05y(2 - x) = 0$, we have horizontal tangents. When $y = 0$, the flow goes to the right. If $x = 2$, the flow goes to the right if $y < 2$ and to the left if $y > 2$. See p.5.

(d) In the quadrant $0 < x < 2, 0 < y < 2$ the general direction of the trajectories is positive for both coordinates.

In $2 < x, 0 < y < 2$, negative in y and positive in x.

In $2 < x, 2 < y$, negative in y and negative in x.

In $0 < x < 2, 2 < y$, positive in y and negative in x. See p.5.

(e) If $x = 0$, y increases without bound. At $x = 0$ (y-axis) arrows point vertically up. If $y = 0$, x increases without bound. At $y = 0$ (x-axis) arrows point horizontally right.

(f) See p.5.

(g) (i) Beast X increases and Beast Y starts steady and then decreases.
(ii) Beast X decreases and beast Y starts steady and then increases.
(iii) Beast X starts steady and then increases, Beast Y decreases.

(h) Supports Darwin because in the long run one of the species wins.

15. (a) $x =$ prey, $y =$predator because the equation for the change in the prey changes with a negative term of y. So the more predators, the fewer the prey.

(b) With no predators, the prey will increase until it hits its the carrying capacity. With no prey, the predators would decrease until 0.

(c) By plug in $(x, y) = (\frac{d}{e}, \frac{a}{c} - \frac{bd}{ce})$,

$$\begin{aligned}
\frac{dx}{dt} &= ax - bx^2 - cxy \\
&= \frac{ad}{e} - \frac{bd^2}{e^2} - \frac{cd}{e}\left(\frac{a}{c} - \frac{bd}{ce}\right) \\
&= \frac{ade - bd^2 - ade + bd^2}{e^2} \\
&= 0\ ; \\
\frac{dy}{dt} &= -dy + exy \\
&= -\frac{da}{c} + \frac{bd^2}{ce} + d\left(\frac{a}{c} - \frac{bd}{ce}\right) \\
&= \frac{-ade + bd^2 + ade - bd^2}{ce} \\
&= 0.
\end{aligned}$$

Thus the point is an equilibrium.

(d) a is the growth rate of x.
b measures the competition among x itself or it contains the information of the carrying capacity of x.
c means the rate of dying out of x due to the predator y.
d is the dying out rate due to no food for the predator y.
e the advantage of y from the prey x.

17. (a) The number of aphids will increase until 100,000. See p.6.

(b) The ladybugs will decrease until 0. See p.6.

(c) See p.???. The nullclines are given by $0 = A(100,000 - A - 50L)$, i.e. $A = 0$ or $A + 50L = 100,000$ and $0 = L(-1000 - L + \frac{1}{25}A)$, i.e.$L = 0$ or $A - 25L = 25,000$.

18. (a) It is a predator-prey system. X is the predator and Y is the prey.

(b) See p.7. The equilibriums are (0,0) and the common solution of

$$\begin{aligned}
-0.1 - 0.1x + y &= 0 \\
1 - 0.1y - x &= 0,
\end{aligned}$$

which is $\left(\frac{99}{101}, \frac{20}{101}\right)$.

(c) Species X remains with no animals and the prey Y increases until 10, its carrying capacity. If both initial conditions are positive, the numbers of X and Y spiral until, in the long run, they get to the equilibrium of $x = \frac{99}{101}$ and $y = \frac{20}{101}$.

13 (a). $\begin{cases} \dfrac{dx}{dt} = -y \\ \dfrac{dy}{dt} = x. \end{cases}$

nullclines :

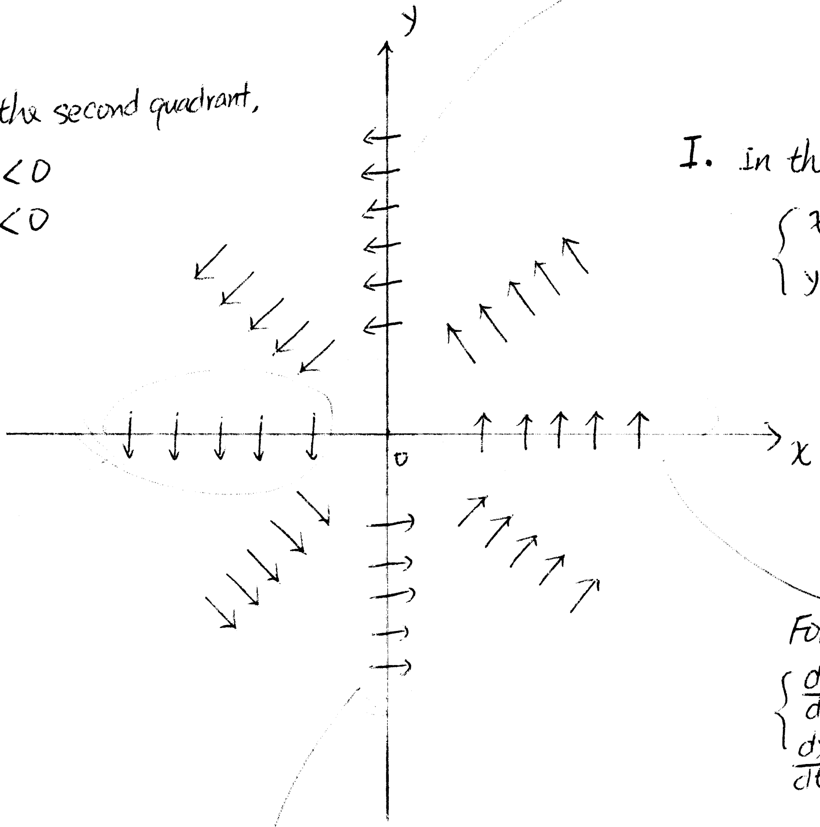
i) If $\frac{dx}{dt} = 0$, i.e. $y = 0$, there will be no horizontal change.
i.e. we have vertical tangents there.

ii) If $\frac{dy}{dt} = 0$, i.e. $x = 0$, there will be no vertical change.
i.e. we have horizontal tangents there.

For $x=0$, $y>0$, $\begin{cases} x' = -y < 0 \\ y' = 0 \end{cases}$

II. In the second quadrant, $\begin{cases} x' < 0 \\ y' < 0 \end{cases}$

I. In the first quadrant, $\begin{cases} x' = -y < 0 \\ y' = x > 0. \end{cases}$



For $x<0$, $y=0$
$\begin{cases} x' = 0 \\ y' = x < 0 \end{cases}$

For $x>0$, $y=0$,
$\begin{cases} \dfrac{dx}{dt} = 0 \\ \dfrac{dy}{dt} = x > 0. \end{cases}$

For $x=0$, $y<0$,
$\begin{cases} x' = -y > 0 \\ y' = 0 \end{cases}$

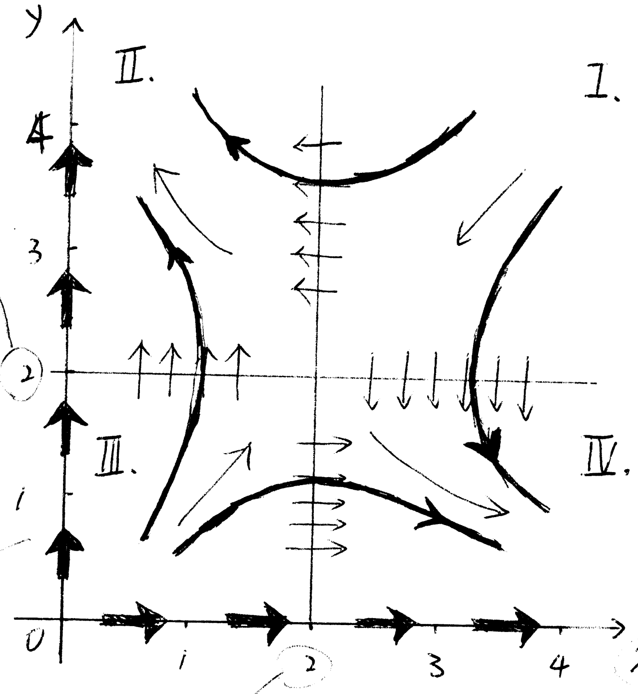
14 (c), (d). $\begin{cases} x' = 0.05x(2-y) \\ y' = 0.05y(2-x). \end{cases}$

- nullclines / vertical - horizontal tangents :

i) If $x'=0$, i.e. $0.05x(2-y)=0$ i.e. $x=0$ or $y=2$,
there is no change of x, i.e. vertical tangents.

ii) If $y'=0$, i.e. $0.05y(2-x)=0$ i.e. $y=0$ or $x=2$,
there is no change of y, i.e. horizontal tangents.

When $y=2$,
$\begin{cases} x'=0 \\ y'=0.1(2-x). \end{cases}$
Then
$y' \begin{cases} >0 \text{ if } x<2 \\ <0 \text{ if } x>2. \end{cases}$



I. In this region,
$\begin{cases} x'<0 \\ y'<0. \end{cases}$

II. $\begin{cases} x'<0 \\ y'>0 \end{cases}$

III. $\begin{cases} x'>0 \\ y'>0 \end{cases}$

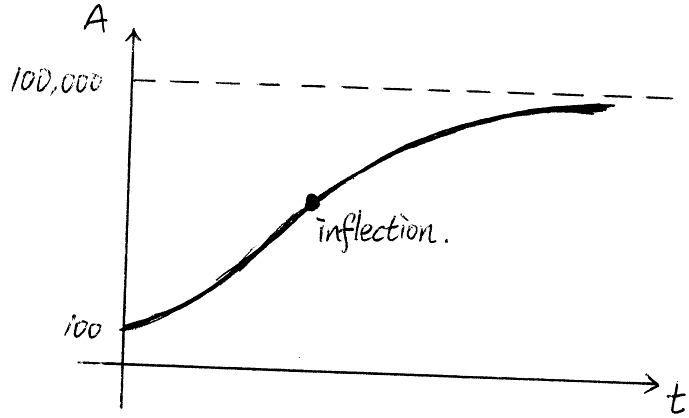
IV. $\begin{cases} x'>0 \\ y'<0. \end{cases}$

When $x=0$,
$\begin{cases} x'=0 \\ y'=0.1y>0. \end{cases}$

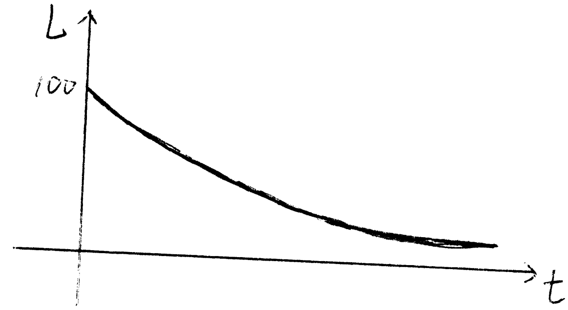
When $x=2$,
$\begin{cases} x'=0.1(2-y) \begin{cases} >0 \text{ if } y<2 \\ <0 \text{ if } y>2. \end{cases} \\ y'=0 \end{cases}$

When $y=0$,
$\begin{cases} x'=0.1x>0 \\ y'=0. \end{cases}$

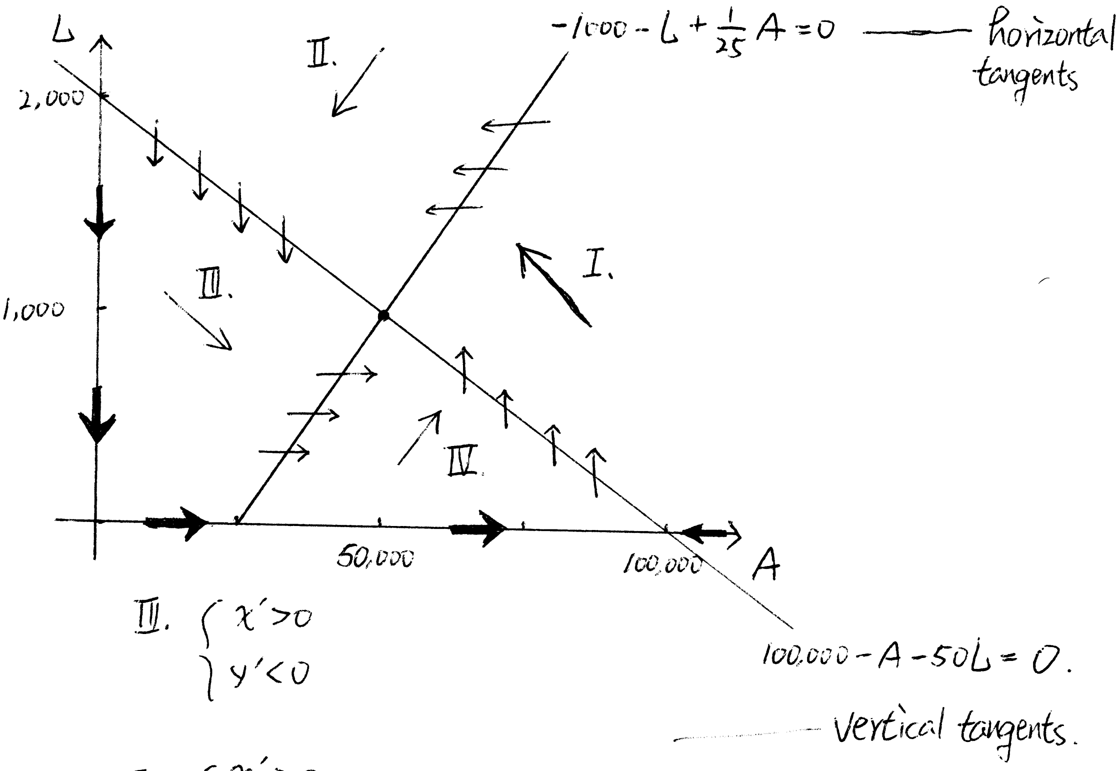
17. a). $\frac{dA}{dt} = A(100{,}000 - A)$



b) $\frac{dL}{dt} = L(-1000 - L)$



c). $\begin{cases} A' = A(100{,}000 - A - 50L) \\ L' = L(-1000 - L + \frac{1}{25}A) \end{cases}$



I. $\begin{cases} x' < 0 \\ y' > 0 \end{cases}$ III. $\begin{cases} x' > 0 \\ y' < 0 \end{cases}$

II. $\begin{cases} x' < 0 \\ y' < 0 \end{cases}$ IV. $\begin{cases} x' > 0 \\ y' > 0. \end{cases}$

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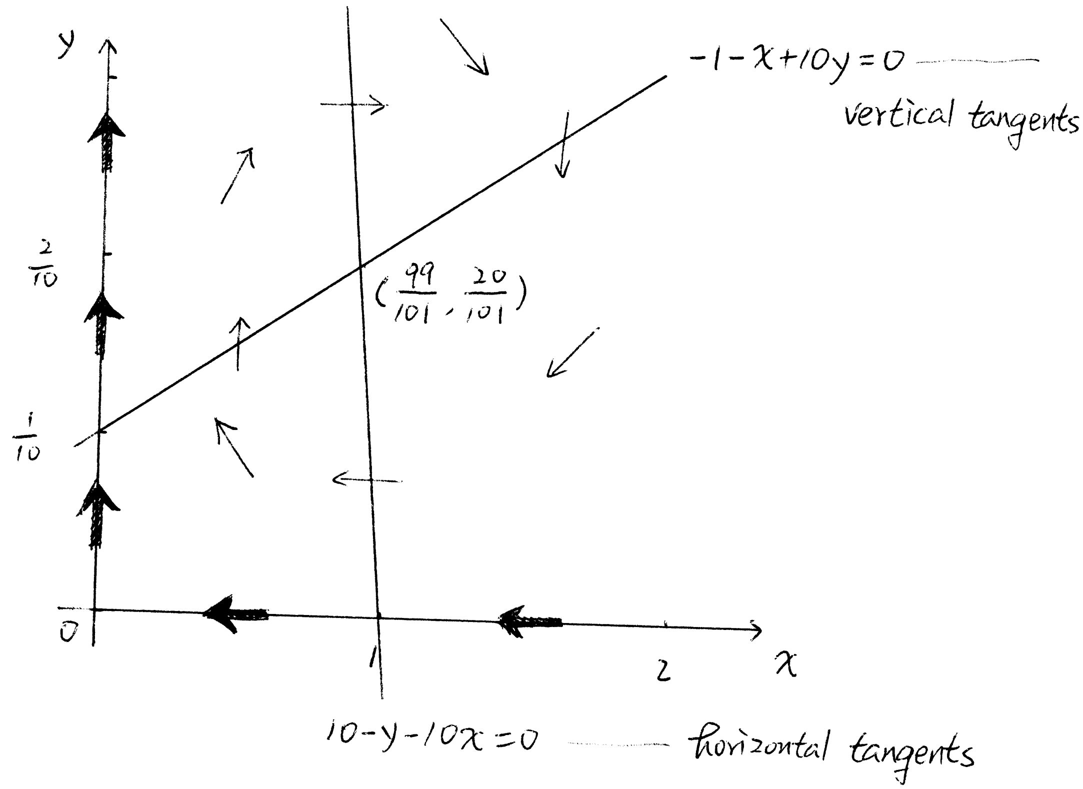
18. b). $\begin{cases} x' = 0.1x(-1-x+10y) \\ y' = 0.1y(10-y-10x) \end{cases}$



- On the line $-1-x+10y=0$, $\frac{dx}{dt}=0$.
  Over this line, $\frac{dx}{dt}>0$; under this line, $\frac{dx}{dt}<0$.
- On the line $10-y-10x=0$, $\frac{dy}{dt}=0$.
  On the right of this line, $\frac{dy}{dt}<0$;
  On the left of this line $\frac{dy}{dt}>0$.