Differential Equations Handout A

21. (a) $y'' + 6y' - 7y = 0,.$

$$r^2 + 6r - 7 = 0,$$

$$(r-1)(r+7) = 0,$$

$$r = 1,\ -7.$$

So the general solution is $y(t) = Ae^t + Be^{-7t}$.

(b) $y'' + 6y' + 9y = 0.$

$$r^2 + 6r + 9 = 0,$$

$$(r+3)^2 = 0,$$

$$r = -3.$$

So the general solution is $y(t) = Ae^{-3t} + Bte^{-3t}$.

(c) $y'' + 5y'z + 6y = 0.$

$$r^2 + 5r + 6 = 0,$$

$$(r+2)(r+3) = 0,$$

$$r = -2,\ -3.$$

So the general solution is $y(t) = Ae^{-2t} + Be^{-3t}$.

22. (a) The general solution is $y(t) = Ae^t + Be^{-7t}$ with $y'(t) = Ae^t - 7Be^{-7t}$. Thus

$$\begin{aligned} A + B &= -2 \\ A - 7B &= 0. \end{aligned}$$

So

$$A = -\frac{7}{4},\ B = -\frac{1}{4}$$

and

$$y(t) = -\frac{7}{4}e^t - \frac{1}{4}e^{-7t}.$$

$$\lim_{t\to\infty} y(t) = -\infty$$

since $\lim_{t\to\infty} e^t = \infty$ and $\lim_{t\to\infty} e^{-7t} = 0$.

(b) $y(t) = Ae^{-3t} + Bte^{-3t}$ and $y'(t) = -3Ae^{-3t} + B(e^{-3t} - 3te^{-3t})$. Thus

$$\begin{aligned} A &= -2 \\ -3A + B &= 0. \end{aligned}$$

So

$$A = -2,\ B = -6$$

and

$$y(t) = -2e^{-3t} - 6te^{-3t}.$$

$$\lim_{t\to\infty} y(t) = 0$$

as $\lim_{t\to\infty} e^{-3t}$ (and te^{-3t}) $= 0$.

(c) $y(t) = Ae^{-2t} + Be^{-3t}$ and $y'(t) = -2Ae^{-2t} - 3Be^{-3t}$.

$$\begin{aligned} A + B &= -2 \\ -2A - 3B &= 0. \end{aligned}$$

So

$$A = -6, \; B = 4$$

and

$$y(t) = -6e^{-2t} + 4e^{-3t}.$$

As one can see easily,

$$\lim_{t \to \infty} y(t) = 0.$$

23. (a) Want to solve

$$x'' + 4x' + 3x = 0$$

with

$$\begin{aligned} x(0) &= 1, \\ x'(0) &= 2. \end{aligned}$$

$$r^2 + 4r + 3 = 0,$$

$$(r+1)(r+3) = 0,$$

$$r = -1, \; -3.$$

The general solution is

$$x(t) = Ae^{-t} + Be^{-3t},$$

and

$$x'(t) = -Ae^{-t} - 3Be^{-3t}.$$

Thus

$$\begin{aligned} A + B &= 1, \\ -A - 3B &= 2. \end{aligned}$$

So

$$A = \frac{5}{2}, \; B = -\frac{3}{2}$$

and the position is

$$x(t) = \frac{5}{2}e^{-t} - \frac{3}{2}e^{-3t}.$$

(b)

$$\begin{aligned} & -\tfrac{3}{2}e^{-3t} + \tfrac{5}{2}e^{-t} &= 0 \\ \leftrightarrow \quad & -3e^{-3t} + 5e^{-t} &= 0 \\ \leftrightarrow \quad & 5e^{-t} &= 3e^{-3t} \\ \leftrightarrow \quad & \tfrac{e^{t}}{e^{-3t}} &= \tfrac{3}{5} \\ \leftrightarrow \quad & e^{2t} &= \tfrac{3}{5} \\ \leftrightarrow \quad & t &= \tfrac{1}{2}\ln\tfrac{3}{5} < 0. \end{aligned}$$

So, no, the mass never crosses.

(c) To find out the critical point, consider

$$x'(t) = -\frac{5}{2}e^{-t} + \frac{9}{2}e^{-3t} = 0 \leftrightarrow 9e^{-3t} = 5e^{-t} \leftrightarrow \frac{9}{5} = \frac{e^{-t}}{e^{-3t}} = e^{2t}.$$

So

$$t = \frac{1}{2}\ln\frac{9}{5}.$$

At the moment,

$$\begin{aligned} x\left(\frac{1}{2}\ln\frac{9}{5}\right) &= \frac{5}{2}e^{\left[-\frac{1}{2}\ln\frac{9}{5}\right]} - \frac{3}{2}e^{\left[-\frac{3}{2}\ln\frac{9}{5}\right]} \\ &= \frac{5}{2}\cdot\sqrt{\frac{5}{9}} - \frac{3}{2}\sqrt{\left(\frac{5}{9}\right)^3} \\ &= \frac{5\sqrt{5}}{6} - \frac{5\sqrt{5}}{18} \\ &\approx 1.25 \end{aligned}$$

26. (a) $y'' - 9y' = 0$.

$$r^2 - 9r = 0 \leftrightarrow r(r-9) = 0 \leftrightarrow r = 0,\ 9.$$

$$y(x) = A + Be^{9x}.$$

(b) $y'' - 9y = 0$.

$$r^2 - 9 = 0 \leftrightarrow (r-3)(r+3) = 0 \leftrightarrow r = -3,\ 3.$$

$$y(x) = Ae^{-3x} + Be^{3x}.$$

(c) $y'' + 9y = 0$.

$$r^2 + 9 = 0 \leftrightarrow r^2 = -9 \leftrightarrow r = \pm 3i.$$

$$y(x) = A\cos 3x + B\sin 3x.$$

(d) $y'' - 9 = 0$. Integrate:

$$y' = 9t + C_1.$$

Integrate again,

$$y = \frac{9}{2}t^2 + C_1 t + C_2.$$

(e) $y'' - 2y' - y = 0$.

$$r^2 - 2r - 1 = 0$$

$$r = \frac{2 \pm \sqrt{4+4}}{2} = \frac{2 \pm 2\sqrt{2}}{2} = 1 \pm \sqrt{2}.$$

$$y(x) = Ae^{(1+\sqrt{2})x} + Be^{(1-\sqrt{2})x}.$$

(f) $y'' - 2y' + 2y = 0$.

$$r^2 - 2r + 2 = 0$$

$$r = \frac{2 \pm \sqrt{4-8}}{2} = \frac{2 \pm \sqrt{-4}}{2} = 1 \pm i.$$

$$y(x) = e^t(A\cos t + B\sin t).$$

27. (a) $x'' = -bx' - cx$. We expect b and c to be positive constants: b is essentially the friction force and c the restoring force. (**Cf.** Supplement p.1045-1046.)

(b) $x'' + bx' + cx = 0$ (with $b, c > 0$). The roots of the characteristic polynomial are

$$\frac{-b \pm \sqrt{b^2 - 4c}}{2}.$$

(i) If $b^2 - 4c > 0$, let

$$\alpha = \frac{-b + \sqrt{b^2 - 4c}}{2}\ ;\ \beta = \frac{-b - \sqrt{b^2 - 4c}}{2}.$$

Then the general solution is

$$x(t) = Ae^{\alpha t} + Be^{\beta t}.$$

Notice in this case, $\alpha, \beta < 0$ since $\sqrt{b^2 - 4c} < b$. Thus $e^{\alpha t}, e^{\beta t} \to 0$ as $t \to \infty$. This implies $x(t) \to 0$ as $t \to \infty$.

(ii) If $b^2 - 4c = 0$, let

$$\gamma = \frac{-b}{2}.$$

Then $\gamma < 0$ and the general solution is

$$x(t) = Ae^{\gamma t} + Bte^{\gamma t}.$$

As in (i), $e^{\gamma t}, te^{\gamma t} \to 0$ as $t \to \infty$. The latter can be seen by applying, e.g. L'Hopital's rule. Thus $\lim_{t\to\infty} x(t) = 0$.

(iii) If $b^2 - 4c < 0$, let

$$\delta = \frac{-b}{2} \ ; \ \epsilon = \frac{\sqrt{4c - b^2}}{2}.$$

Then $\delta < 0$ and the general solution is

$$x(t) = e^{\delta t}(A \cos \epsilon t + B \sin \epsilon t).$$

In this case,

$$|x(t)| \leq e^{\delta t} C \to 0 \qquad \text{as } t \to \infty.$$

Therefore $\lim_{t\to\infty} x(t) = 0$.

28. (a) Want r to be negative for both roots of the characteristic polynomial. So, for example, consider

$$(r + 1)(r + 2) = r^2 + rx + 2.$$

The differential equation

$$x'' + 3x' + 2x = 0$$

with initial conditions

$$\begin{aligned} x(0) &= 1 \\ x'(0) &= 2 \end{aligned}$$

has solution

$$x(t) = 4e^{-t} - 3e^{-2t}.$$

and $\lim_{t\to\infty} x(t) = 0$.

(b) Want r to be positive for at least one of the roots and the corresponding solution $x(t)$ satisfies $x(0) = 1, x'(0) = 2$. Consider

$$(r - 2)(r + 2005811) = r^2 + 2005809r - 4011622.$$

The corresponding

$$x'' + 2005809x' - 4011622x = 0$$

with

$$\begin{aligned} x(0) &= 1 \\ x'(0) &= 2 \end{aligned}$$

has solution

$$x(t) = e^{2t},$$

and $\lim_{t\to\infty} x(t) = \infty$. (**Note that we can replace 2005811 by any real number!**)

(c) If the solution any involves sine and cosine functions, then the limit does not exist because of the oscillation of sine and cosine. This corresponds the the case that the corresponding characteristic polynomial has only purely imaginary roots. For example, consider

$$(r - i)(r + i) = r^2 + 1.$$

The corresponding $x'' + x = 0$ with the initial conditions has solution

$$x(t) = \cos t + 2 \sin t,$$

and the limit does not exist.

Supplement 31.6

12. Consider the corresponding quadratic polynomial

$$z^2 + bz + c = 0$$

with roots

$$z = \frac{-b \pm \sqrt{b^2 - 4c}}{2}.$$

To have a periodic solution corresponds that the quadratic polynomial should have purely imaginary roots. Thus $b = 0$ and $-4c < 0$. I.e. $b = 0$ and $c > 0$. In this case, the general solution is

$$x(t) = A\cos(\sqrt{4c})t + B\sin(\sqrt{4c})t.$$

Hence the period is $\frac{2\pi}{\sqrt{4c}} = \frac{\pi}{\sqrt{c}}$.

14. (i) If $b^2 - 4c > 0$, let

$$\alpha = \frac{-b + \sqrt{b^2 - 4c}}{2} \ ; \ \beta = \frac{-b - \sqrt{b^2 - 4c}}{2}.$$

Then the general solution is

$$x(t) = Ae^{\alpha t} + Be^{\beta t}.$$

Notice in this case, $\alpha, \beta < 0$ since $\sqrt{b^2 - 4c} < b$. Thus $e^{\alpha t}, e^{\beta t} \to 0$ as $t \to \infty$. This implies $x(t) \to 0$ as $t \to \infty$.

(ii) If $b^2 - 4c = 0$, let

$$\gamma = \frac{-b}{2}.$$

Then $\gamma < 0$ and the general solution is

$$x(t) = Ae^{\gamma t} + Bte^{\gamma t}.$$

As in (i), $e^{\gamma t}, te^{\gamma t} \to 0$ as $t \to \infty$. The latter can be seen by applying, e.g. L'Hopital's rule. Thus $\lim_{t\to\infty} x(t) = 0$.

(iii) If $b^2 - 4c < 0$, let

$$\delta = \frac{-b}{2} \ ; \ \epsilon = \frac{\sqrt{4c - b^2}}{2}.$$

Then $\delta < 0$ and the general solution is

$$x(t) = e^{\delta t}(\cos \epsilon t + \sin \epsilon t).$$

In this case,

$$|x(t)| \le e^{\delta t}(1 + 1) \to 0 \qquad \text{as } t \to \infty.$$

Therefore $\lim_{t\to\infty} x(t) = 0$.

(iv) In the case of vibrating a spring, c essentially measures the spring constant and b measures the friction constant. We expect if $b, c > 0$, the spring will eventually converge to a complete stop at the natural length.

16. Let $x(t)$ be the position function of the mass so that $x = 0$ refers to the natural length. The spring constant k is given by the equation

$$10 = (0.6 - 0.5)k.$$

Thus $k = 100$ (N/m). The differential equation modeling the position is

$$2 \cdot \frac{d^2x}{dt^2} = mass \cdot acceleration = force = -kx = -100x,$$

i.e.

$$x''(t) = -50x,$$

with initial conditions

$$\begin{aligned} x(0) &= 0.4 - 0.6 = -0.2 \\ x'(0) &= 0. \end{aligned}$$

The characteristic polynomial is

$$z^2 + 50 = 0$$

with roots

$$z = \pm\sqrt{-50} = \pm 5\sqrt{2}i.$$

Hence the general solution is

$$x(t) = A\cos 5\sqrt{2}t + B\sin 5\sqrt{2}t,$$

with derivative

$$x'(t) = -5\sqrt{2}A\sin 5\sqrt{2}t + 5\sqrt{2}B\cos 5\sqrt{2}t.$$

Plug in the initial conditions, we have

$$\begin{aligned} A &= -0.2 \\ 5\sqrt{2}B &= 0. \end{aligned}$$

Therefore the position of the mass is

$$x(t) = -0.2\cos 5\sqrt{2}t.$$

17. We want to find constants a, b, c such that

$$ay'' + by' + cy = 0.$$

For $y(t) = e^t \sin t$,

$$\begin{aligned} y'(t) &= e^t \sin t + e^t \cos t \\ &= e^t(\sin t + \cos t), \\ y''(t) &= e^t \sin t + e^t \cos t + e^t \cos t - e^t \sin t \\ &= 2e^t \cos t. \end{aligned}$$

In this case, we can take, for example, $a = 1, b = -2, c = 2$ or any a, b, c such that

$$2a + b = 0, \;\; b + c = 0.$$