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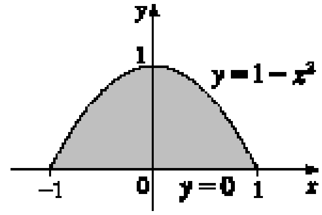
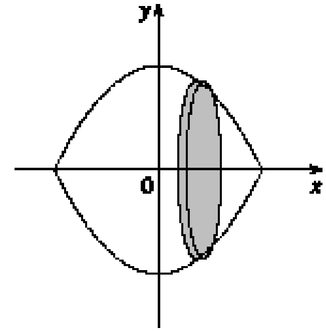
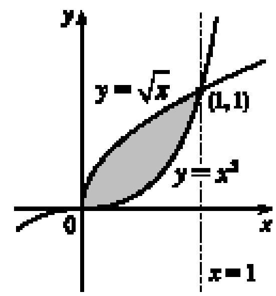
2. A cross-section is a disk with radius $1 - x^2$, so its area is $A(x) = \pi(1 - x^2)^2$.





$$V = \int_{-1}^{1} A(x)\,dx = \int_{-1}^{1} \pi(1 - x^2)^2\,dx$$
$$= 2\pi \int_0^1 \left(1 - 2x^2 + x^4\right) dx$$
$$= 2\pi\left[x - \tfrac{2}{3}x^3 + \tfrac{1}{5}x^5\right]_0^1 = 2\pi\left(1 - \tfrac{2}{3} + \tfrac{1}{5}\right)$$
$$= 2\pi\left(\tfrac{8}{15}\right) = \tfrac{16\pi}{15}$$

13. $y = \sqrt{x} \;\Rightarrow\; x = y^2$ and $y = x^3 \;\Rightarrow\; x = \sqrt[3]{y}$. A cross-section is a washer with inner radius $1 - \sqrt[3]{y}$ and outer radius $1 - y^2$, so its area is



$A(y) = \pi(1 - y^2)^2 - \pi\left(1 - \sqrt[3]{y}\right)^2$.

$$V = \int_0^1 A(y)\,dy = \int_0^1 \left[\pi(1 - y^2)^2 - \pi\left(1 - \sqrt[3]{y}\right)^2\right] dy$$
$$= \pi \int_0^1 [(1 - 2y^2 + y^4) - (1 - 2y^{1/3} + y^{2/3})]\,dy$$
$$= \pi \int_0^1 (-2y^2 + y^4 + 2y^{1/3} - y^{2/3})\,dy = \pi\left[-\tfrac{2}{3}y^3 + \tfrac{1}{5}y^5 + \tfrac{3}{2}y^{4/3} - \tfrac{3}{5}y^{5/3}\right]_0^1$$
$$= \pi\left(-\tfrac{2}{3} + \tfrac{1}{5} + \tfrac{3}{2} - \tfrac{3}{5}\right) = \tfrac{13\pi}{30}$$

14. A cross-section is a washer with inner radius $1 - \sqrt{x}$ and outer radius $1 - x^3$, so its area is $A(x) = \pi(1 - x^3)^2 - \pi(1 - \sqrt{x})^2$.

$$V = \int_0^1 A(x)\,dx = \int_0^1 \left[\pi(1 - x^3)^2 - \pi\left(1 - \sqrt{x}\right)^2\right] dx = \pi \int_0^1 [(1 - 2x^3 + x^6) - (1 - 2x^{1/2} + x)]\,dx$$
$$= \pi \int_0^1 (-2x^3 + x^6 + 2x^{1/2} - x)\,dx = \pi\left[-\tfrac{1}{2}x^4 + \tfrac{1}{7}x^7 + \tfrac{4}{3}x^{3/2} - \tfrac{1}{2}x^2\right]_0^1 = \pi\left(-\tfrac{1}{2} + \tfrac{1}{7} + \tfrac{4}{3} - \tfrac{1}{2}\right) = \tfrac{10\pi}{21}$$

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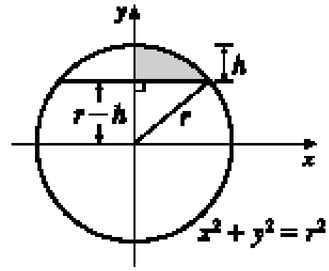
27. $x^2 + y^2 = r^2 \quad \Leftrightarrow \quad x^2 = r^2 - y^2$



$$V = \pi \int_{r-h}^{r} \left(r^2 - y^2\right) dy = \pi \left[r^2 y - \frac{y^3}{3} \right]_{r-h}^{r}$$

$$= \pi \left\{ \left[r^3 - \frac{r^3}{3} \right] - \left[r^2(r-h) - \frac{(r-h)^3}{3} \right] \right\}$$

$$= \pi \left\{ \tfrac{2}{3} r^3 - \tfrac{1}{3}(r-h)\left[3r^2 - (r-h)^2\right] \right\}$$

$$= \tfrac{1}{3}\pi \left\{ 2r^3 - (r-h)\left[3r^2 - \left(r^2 - 2rh + h^2\right)\right] \right\}$$

$$= \tfrac{1}{3}\pi \left\{ 2r^3 - (r-h)\left[2r^2 + 2rh - h^2\right] \right\}$$

$$= \tfrac{1}{3}\pi \left(2r^3 - 2r^3 - 2r^2 h + rh^2 + 2r^2 h + 2rh^2 - h^3\right)$$

$$= \tfrac{1}{3}\pi \left(3rh^2 - h^3\right) = \tfrac{1}{3}\pi h^2 (3r - h), \text{ or, equivalently, } \pi h^2 \left(r - \frac{h}{3}\right)$$

46. The line $y = r$ intersects the semicircle $y = \sqrt{R^2 - x^2}$ when $r = \sqrt{R^2 - x^2} \quad \Rightarrow \quad r^2 = R^2 - x^2 \quad \Rightarrow$ $x^2 = R^2 - r^2 \quad \Rightarrow \quad x = \pm\sqrt{R^2 - r^2}$. Rotating the shaded region about the x-axis gives us

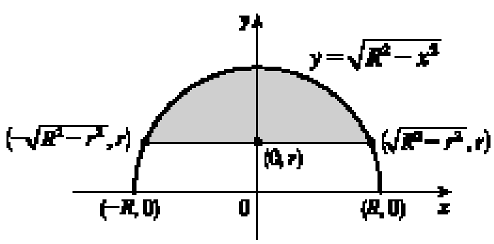
$$V = \int_{-\sqrt{R^2-r^2}}^{\sqrt{R^2-r^2}} \pi \left[\left(\sqrt{R^2 - x^2}\right)^2 - r^2 \right] dx = 2\pi \int_0^{\sqrt{R^2-r^2}} \left(R^2 - x^2 - r^2\right) dx \qquad \text{[by symmetry]}$$

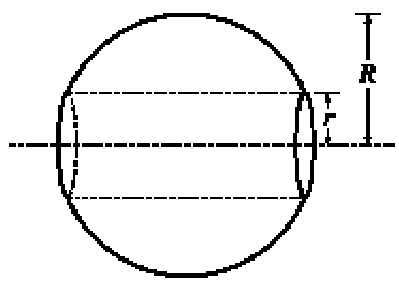
$$= 2\pi \int_0^{\sqrt{R^2-r^2}} \left[\left(R^2 - r^2\right) - x^2 \right] dx = 2\pi \left[\left(R^2 - r^2\right)x - \tfrac{1}{3}x^3 \right]_0^{\sqrt{R^2-r^2}}$$

$$= 2\pi \left[\left(R^2 - r^2\right)^{3/2} - \tfrac{1}{3}\left(R^2 - r^2\right)^{3/2} \right] = 2\pi \cdot \tfrac{2}{3}\left(R^2 - r^2\right)^{3/2} = \tfrac{4\pi}{3}\left(R^2 - r^2\right)^{3/2}$$

Our answer makes sense in limiting cases. As $r \to 0$, $V \to \frac{4}{3}\pi R^3$, which is the volume of the full sphere. As $r \to R$, $V \to 0$, which makes sense because the hole's radius is approaching that of the sphere

.





1. $y = 2 - 3x \quad \Rightarrow \quad L = \int_{-2}^{1} \sqrt{1 + (dy/dx)^2}\, dx = \int_{-2}^{1} \sqrt{1 + (-3)^2}\, dx = \sqrt{10}\, [1 - (-2)] = 3\sqrt{10}$.

The arc length can be calculated using the distance formula, since the curve is a line segment, so

$$L = [\text{distance from } (-2, 8) \text{ to } (1, -1)] = \sqrt{[1 - (-2)]^2 + [(-1) - 8]^2} = \sqrt{90} = 3\sqrt{10}$$

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3. $y = \cos x \quad \Rightarrow \quad dy/dx = -\sin x \quad \Rightarrow \quad 1 + (dy/dx)^2 = 1 + \sin^2 x$. So $L = \int_0^{2\pi} \sqrt{1 + \sin^2 x}\, dx$.

4. $$h_{\text{ave}} = \frac{1}{6-1}\int_1^6 \frac{3}{(1+r)^2}\,dr = \frac{1}{5}\int_2^7 3u^{-2}\,du \qquad [u = 1 + r, du = dr]$$
$$= -\tfrac{3}{5}\left[u^{-1}\right]_2^7 = -\tfrac{3}{5}\left(\tfrac{1}{7} - \tfrac{1}{2}\right) = \tfrac{3}{5}\left(\tfrac{1}{2} - \tfrac{1}{7}\right) = \tfrac{3}{5} \cdot \tfrac{5}{14} = \tfrac{3}{14}$$

12. (a) $v_{\text{ave}} = \frac{1}{12-0}\int_0^{12} v(t)\,dt = \frac{1}{12}I$. Use the Midpoint Rule with $n = 3$ and $\Delta t = \frac{12-0}{3} = 4$ to estimate I.
$I \approx M_3 = 4[v(2) + v(6) + v(10)] = 4[21 + 50 + 66] = 4(137) = 548$. Thus, $v_{\text{ave}} \approx \frac{1}{12}(548) = 45\frac{2}{3}$ km/h.

(b) Estimating from the graph, $v(t) = 45\frac{2}{3}$ when $t \approx 5.2$ s.

Integration Handout B

4

(a) $\pi \int_0^4 y \ dy$
(b) $\pi \int_0^4 (\sqrt{y}+2)^2 - (2-\sqrt{y})^2 \ dy$
(c) $\pi \int_{-2}^2 (4-x^2)^2 \ dx$
(d) $\pi \int_{-2}^2 25-(1+x^2)^2 \ dy$

5 First find the total volume

$$V = \int_0^8 \pi(\sqrt[3]{y})^2 \ dy = \pi\Big[\frac{3}{5}y^{\frac{5}{3}}\Big]_0^8 = \frac{96}{5}\pi.$$

So $\frac{1}{2}$ the volume is $V = \frac{48}{5}\pi$.

$$\begin{aligned} \frac{48}{5}\pi &= \int_0^h \pi y^{\frac{2}{3}} \ dy \\ &= \pi\Big[\frac{3}{5}y^{\frac{5}{3}}\Big]_0^h \\ &= \pi\frac{3}{5}h^{\frac{5}{3}} = \frac{48}{5}\pi \\ 16 &= h^{\frac{5}{3}} \\ h &= 16^{\frac{3}{5}}. \end{aligned}$$

Stop drinking when the height is $16^{3/5}$.

12 $\int_0^{10} 4\pi x^2 \rho(x) \ dx$.

13 (a) $\int_0^R 4\pi x^2 \rho(x) \ dx$.

(b) $\int_0^R \pi(R^2-x^2)\delta(x) \ dx$.

14 1. This integral is similar to the one above with $x := h$ and $R := 100$ and $\delta(x) := 6 \ x \ 10^{-45}(200-h)$ grams per cubic foot. The integral and answer is then:

$$\int_0^{100} \pi(100^2-h^2)6\cdot 10^{-5}(200-h) \ dh$$

2. Evaluating this integral:

$$\begin{aligned} &\int_0^{100} \pi(100^2-h^2)6\cdot 10^{-5}(200-h) \ dh \\ &= 6\pi\cdot 10^{-5}\int_0^{100} (10^4-h^2)(200-h) \ dh \\ &= 6\pi\cdot 10^{-5}\int_0^{100} (2\cdot 10^6 - h10^4 - 200h^2 + h^3) \ dh = 6500\pi \end{aligned}$$

15 We should superimpose the graphs of the functions a), b) and c). Since the interval is the same the largest area will represent the greatest average value.

From visual inspection one can conclude that the greatest average value in the interval $[-1, 1]$ is that of $y = \sqrt{1 - x^2}$ ($\sim .75$), followed by $y = \exp^{-|x|}$ ($\sim .63$) and the function with smallest average value is $y = -|x| + 1$ ($\sim .5$).

18 If January corresponds to $t = 0$, $t = -1$ would be December and $t = 1$ would be February. The integral of a rate of change represents the change itself. $\int_{-1}^{1} f(x)\, dx$ is the change of the numbers of zebras in Seronera during the months of December and January.