Name: ___________________ Class:___ Date: _____________

16. (a) $\dfrac{x-1}{x^3+x^2} = \dfrac{x-1}{x^2(x+1)} = \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{x+1}$

(b) $\dfrac{x-1}{x^3+x} = \dfrac{x-1}{x(x^2+1)} = \dfrac{A}{x} + \dfrac{Bx+C}{x^2+1}$

18. $\dfrac{x-1}{x^2+3x+2} = \dfrac{A}{x+1} + \dfrac{B}{x+2}$. Multiply both sides by $(x+1)(x+2)$ to get $x-1 = A(x+2) + B(x+1)$. Substituting -2 for x gives $-3 = -B \quad \Leftrightarrow \quad B = 3$. Substituting -1 for x gives $-2 = A$. Thus,

$$\int_0^1 \frac{x-1}{x^2+3x+2}\,dx = \int_0^1 \left(\frac{-2}{x+1} + \frac{3}{x+2}\right) dx = \left[-2\ln|x+1| + 3\ln|x+2|\right]_0^1$$
$$= (-2\ln 2 + 3\ln 3) - (-2\ln 1 + 3\ln 2) = 3\ln 3 - 5\ln 2 \quad \left[\text{or } \ln\tfrac{27}{32}\right]$$

20. $\dfrac{x^2+2x-1}{x^3-x} = \dfrac{x^2+2x-1}{x(x+1)(x-1)} = \dfrac{A}{x} + \dfrac{B}{x+1} + \dfrac{C}{x-1}$. Multiply both sides by $x(x+1)(x-1)$ to get $x^2+2x-1 = A(x+1)(x-1) + Bx(x-1) + Cx(x+1)$. Substituting 0 for x gives $-1 = -A \quad \Leftrightarrow \quad A = 1$. Substituting -1 for x gives $-2 = 2B \quad \Leftrightarrow \quad B = -1$. Substituting 1 for x gives $2 = 2C \quad \Leftrightarrow \quad C = 1$. Thus,
$\displaystyle\int \frac{x^2+2x-1}{x^3-x}\,dx = \int\left(\frac{1}{x} - \frac{1}{x+1} + \frac{1}{x-1}\right)dx = \ln|x| - \ln|x+1| + \ln|x-1| + C = \ln\left|\frac{x(x-1)}{x+1}\right| + C.$

24. $\dfrac{x^2-x+6}{x^3+3x} = \dfrac{x^2-x+6}{x(x^2+3)} = \dfrac{A}{x} + \dfrac{Bx+C}{x^2+3}$. Multiply by $x(x^2+3)$ to get $x^2-x+6 = A(x^2+3) + (Bx+C)x$. Substituting 0 for x gives $6 = 3A \quad \Leftrightarrow \quad A = 2$. The coefficients of the x^2-terms must be equal, so $1 = A + B \quad \Rightarrow \quad B = 1 - 2 = -1$. The coefficients of the x-terms must be equal, so $-1 = C$. Thus,

$$\int \frac{x^2-x+6}{x^3+3x}\,dx = \int\left(\frac{2}{x} + \frac{-x-1}{x^2+3}\right)dx = \int\left(\frac{2}{x} - \frac{x}{x^2+3} - \frac{1}{x^2+3}\right)dx$$
$$= 2\ln|x| - \frac{1}{2}\ln(x^2+3) - \frac{1}{\sqrt{3}}\tan^{-1}\frac{x}{\sqrt{3}} + C$$

26. $\displaystyle\int \frac{r^2}{r+4}\,dr = \int\left(\frac{r^2-16}{r+4} + \frac{16}{r+4}\right)dr = \int\left(r - 4 + \frac{16}{r+4}\right)dr$ [or use long division]
$= \frac{1}{2}r^2 - 4r + 16\ln|r+4| + C$

29. Let $u = \sqrt{x}$, so $u^2 = x$ and $dx = 2u\,du$. Thus,

$$\int_9^{16} \frac{\sqrt{x}}{x-4}\,dx = \int_3^4 \frac{u}{u^2-4} 2u\,du = 2\int_3^4 \frac{u^2}{u^2-4}\,du = 2\int_3^4 \left(1 + \frac{4}{u^2-4}\right)du \qquad \text{[by long division]}$$

$$= 2 + 8\int_3^4 \frac{du}{(u+2)(u-2)}. \quad (*)$$

Multiply $\dfrac{1}{(u+2)(u-2)} = \dfrac{A}{u+2} + \dfrac{B}{u-2}$ by $(u+2)(u-2)$ to get $1 = A(u-2) + B(u+2)$. Equating coefficients we get $A + B = 0$ and $-2A + 2B = 1$. Solving gives us $B = \frac{1}{4}$ and $A = -\frac{1}{4}$, so $\dfrac{1}{(u+2)(u-2)} = \dfrac{-1/4}{u+2} + \dfrac{1/4}{u-2}$ and $(*)$ is

$$2 + 8\int_3^4 \left(\frac{-1/4}{u+2} + \frac{1/4}{u-2}\right)du = 2 + 8\left[-\tfrac{1}{4}\ln|u+2| + \tfrac{1}{4}\ln|u-2|\right]_3^4$$

$$= 2 + \Big[2\ln|u-2| - 2\ln|u+2|\Big]_3^4 = 2 + 2\left[\ln\left|\frac{u-2}{u+2}\right|\right]_3^4$$

$$= 2 + 2\left(\ln\tfrac{2}{6} - \ln\tfrac{1}{5}\right) = 2 + 2\ln\tfrac{2/6}{1/5}$$

$$= 2 + 2\ln\tfrac{5}{3} \text{ or } 2 + \ln\left(\tfrac{5}{3}\right)^2 = 2 + \ln\tfrac{25}{9}$$

2. (a) Since $y = \dfrac{1}{2x-1}$ is defined and continuous on $[1, 2]$, $\displaystyle\int_1^2 \frac{1}{2x-1}\,dx$ is proper.

(b) Since $y = \dfrac{1}{2x-1}$ has an infinite discontinuity at $x = \frac{1}{2}$, $\displaystyle\int_0^1 \frac{1}{2x-1}\,dx$ is a Type II improper integral.

(c) Since $\displaystyle\int_{-\infty}^{\infty} \frac{\sin x}{1+x^2}\,dx$ has an infinite interval of integration, it is an improper integral of Type I.

(d) Since $y = \ln(x-1)$ has an infinite discontinuity at $x = 1$, $\int_1^2 \ln(x-1)dx$ is a Type II improper integral.

8. $\displaystyle\int_0^\infty \frac{x}{(x^2+2)^2}\,dx = \lim_{t\to\infty}\int_0^t \frac{x}{(x^2+2)^2}\,dx = \lim_{t\to\infty}\frac{1}{2}\left[\frac{-1}{x^2+2}\right]_0^t = \frac{1}{2}\lim_{t\to\infty}\left(\frac{-1}{t^2+2} + \frac{1}{2}\right)$

$= \frac{1}{2}\left(0 + \frac{1}{2}\right) = \frac{1}{4}$. Convergent

14. $\int_{-\infty}^{\infty} x^2 e^{-x^3}\,dx = \int_{-\infty}^{0} x^2 e^{-x^3}\,dx + \int_0^\infty x^2 e^{-x^3}\,dx$, and

$\displaystyle\int_{-\infty}^0 x^2 e^{-x^3}\,dx = \lim_{t\to-\infty}\left[-\tfrac{1}{3}e^{-x^3}\right]_t^0 = -\tfrac{1}{3} + \tfrac{1}{3}\left(\lim_{t\to-\infty} e^{-t^3}\right) = \infty$. Divergent

16. $I = \int_{-\infty}^{\infty} \cos \pi t\,dt = I_1 + I_2 = \int_{-\infty}^0 \cos \pi t\,dt + \int_0^\infty \cos \pi t\,dt$, but $I_1 = \displaystyle\lim_{s\to-\infty}\left[\frac{1}{\pi}\sin \pi t\right]_s^0 = \lim_{s\to-\infty}\left(-\frac{1}{\pi}\sin \pi t\right)$ and this limit does not exist. Since I_1 is divergent, I is divergent, and there is no need to evaluate I_2. Divergent

24. $\int_2^3 \frac{1}{\sqrt{3-x}}\,dx = \lim_{t\to 3^-} \int_2^t (3-x)^{-1/2}\,dx = \lim_{t\to 3^-} \left[-2(3-x)^{1/2}\right]_2^t = -2\lim_{t\to 3^-}\left(\sqrt{3-t}-\sqrt{1}\right)$

$= -2(0-1) = 2.$ Convergent

29. There is an infinite discontinuity at $x = 0$. $\int_{-1}^1 \frac{e^x}{e^x-1}\,dx = \int_{-1}^0 \frac{e^x}{e^x-1}\,dx + \int_0^1 \frac{e^x}{e^x-1}\,dx.$

$\int_{-1}^0 \frac{e^x}{e^x-1}\,dx = \lim_{t\to 0^-}\int_{-1}^t \frac{e^x}{e^x-1}\,dx = \lim_{t\to 0^-}\left[\ln\left|e^x-1\right|\right]_{-1}^t = \lim_{t\to 0^-}\left[\ln\left|e^t-1\right| - \ln\left|e^{-1}-1\right|\right] = -\infty,$

so $\int_{-1}^1 \frac{e^x}{e^x-1}\,dx$ is divergent. The integral $\int_0^1 \frac{e^x}{e^x-1}\,dx$ also diverges since

$\int_0^1 \frac{e^x}{e^x-1}\,dx = \lim_{t\to 0^+}\int_t^1 \frac{e^x}{e^x-1}\,dx = \lim_{t\to 0^+}\left[\ln\left|e^x-1\right|\right]_t^1 = \lim_{t\to 0^+}\left[\ln|e-1| - \ln\left|e^t-1\right|\right] = \infty.$

Divergent

44. For $x \geq 1$, $0 < \frac{x}{\sqrt{1+x^6}} < \frac{x}{\sqrt{x^6}} = \frac{x}{x^3} = \frac{1}{x^2}$. $\int_1^\infty \frac{1}{x^2}\,dx$ is convergent by Equation 2 with $p = 2 > 1$, so

$\int_1^\infty \frac{x}{\sqrt{1+x^6}}\,dx$ is convergent by the Comparison Theorem.

50. (a) $\boldsymbol{n=0}$: $\int_0^\infty x^n e^{-x}\,dx = \lim_{t\to\infty}\int_0^t e^{-x}\,dx = \lim_{t\to\infty}\left[-e^{-x}\right]_0^t$

$= \lim_{t\to\infty}\left[-e^{-t}+1\right] = 0+1 = 1$

$\boldsymbol{n=1}$: $\int_0^\infty x^n e^{-x}\,dx = \lim_{t\to\infty}\int_0^t xe^{-x}\,dx$. To evaluate $\int xe^{-x}\,dx$, we'll use integration by parts with $u=x$, $dv=e^{-x}dx \;\Rightarrow\; du=dx$, $v=-e^{-x}$.

So $\int xe^{-x}\,dx = -xe^{-x} - \int -e^{-x}\,dx = -xe^{-x} - e^{-x} + C = (-x-1)e^{-x}+C$ and

$\lim_{t\to\infty}\int_0^t xe^{-x}\,dx = \lim_{t\to\infty}\left[(-x-1)e^{-x}\right]_0^t$

$= \lim_{t\to\infty}\left[(-t-1)e^{-t}+1\right] = \lim_{t\to\infty}\left[-te^{-t}-e^{-t}+1\right]$

$= 0-0+1$ [use l'Hospital's Rule] $=1$

$\boldsymbol{n=2}$: $\int_0^\infty x^n e^{-x}\,dx = \lim_{t\to\infty}\int_0^t x^2e^{-x}\,dx$. To evaluate $\int x^2e^{-x}\,dx$, we could use integration by parts again or Formula 97. Thus,

$\lim_{t\to\infty}\int_0^t x^2e^{-x}\,dx = \lim_{t\to\infty}\left[-x^2e^{-x}\right]_0^t + 2\lim_{t\to\infty}\int_0^t xe^{-x}\,dx$

$= 0+0+2(1)$ [use l'Hospital's Rule and the result for $n=1$] $=2$

$\boldsymbol{n=3}$: $\int_0^\infty x^n e^{-x}\,dx = \lim_{t\to\infty}\int_0^t x^3e^{-x}\,dx \overset{97}{=} \lim_{t\to\infty}\left[-x^3e^{-x}\right]_0^t + 3\lim_{t\to\infty}\int_0^t x^2e^{-x}\,dx$

$= 0+0+3(2)$ [use l'Hospital's Rule and the result for $n=2$] $=6$

(b) For $n=1$, 2, and 3, we have $\int_0^\infty x^n e^{-x}\,dx = 1$, 2, and 6. The values for the integral are equal to the factorials for n, so we guess $\int_0^\infty x^n e^{-x}\,dx = n!$. For $n=1$, 2, and 3, we have $\int_0^\infty x^n e^{-x}\,dx = 1$, 2, and 6. The values for the integral are equal to the factorials for n, so we guess $\int_0^\infty x^n e^{-x}\,dx = n!$.

(c) Suppose that $\int_0^\infty x^k e^{-x}\,dx = k!$ for some positive integer k. Then $\int_0^\infty x^{k+1}e^{-x}\,dx = \lim_{t\to\infty}\int_0^t x^{k+1}e^{-x}\,dx$. To evaluate $\int x^{k+1}e^{-x}\,dx$, we use parts with $u=x^{k+1}$, $dv=e^{-x}dx \;\Rightarrow\; du=(k+1)x^k dx$, $v=-e^{-x}$.

So $\int x^{k+1}e^{-x}\,dx = -x^{k+1}e^{-x} - \int -(k+1)x^ke^{-x}\,dx = -x^{k+1}e^{-x} + (k+1)\int x^ke^{-x}\,dx$ and

$\lim_{t\to\infty}\int_0^t x^{k+1}e^{-x}\,dx = \lim_{t\to\infty}\left[-x^{k+1}e^{-x}\right]_0^t + (k+1)\lim_{t\to\infty}\int_0^t x^ke^{-x}\,dx$

$= \lim_{t\to\infty}\left[-t^{k+1}e^{-t}+0\right] + (k+1)k! = 0+0+(k+1)! = (k+1)!$,

so the formula holds for $k+1$. By induction, the formula holds for all positive integers. (Since $0!=1$, the formula holds for $n=0$, too.)

Integration Handout B

17 First we fix one "coordinate system"! Let x be the horizontal distance from the base of the tank, which is a disk of radius 5m, i.e. the axis is a vertical line with positive direction being pointed up and the grand is the origin. When fix an x, the mass of the water of the horizontal slice at position x with height Δx is given by

$$\pi(\sqrt{5^2 - x^2})^2 \cdot \Delta x \cdot 100.$$

Thus the total work would be approximately

$$\sum \pi(\sqrt{5^2 - x^2})^2 \cdot \Delta x \cdot 100 \cdot 9.8 \cdot [2 + (5 - x)].$$

Taking the limit, we have

$$\begin{aligned} W &= \int_0^5 \pi(25 - x^2) \cdot 100 \cdot 9.8 \cdot (7 - x)\ dx \\ &= 980\pi \int_0^5 175 - 25x - 7x^2 + x^3\ dx \\ &= 980\pi[175x - \tfrac{25}{2}x^2 - \tfrac{7}{3}x^3 + \tfrac{1}{4}x^4]_0^5 \\ &= \tfrac{5^5 \cdot 7^2 \cdot 41}{3}\pi. \end{aligned}$$

20

(a)

$$\int \sin^2 x\ \cos^3 x\ dx = \int \sin^2 x\ \cos^2 x\ \cos x\ dx$$

$$= \int \sin^2 x\ (1 - \sin^2 x)\ \cos x\ dx = \int \sin^2 x\ \cos x\ dx - \int \sin^4 x\ \cos x\ dx$$

$$= \frac{\sin^3 x}{3} - \frac{\sin^5 x}{5} + C$$

(b)

$$\int \sin^5 x\ \cos^4 x\ dx = \int \sin x\ \sin^4 x\ \cos^4 x\ dx$$

$$= \int \sin x\ (1 - \cos^2 x)^2\ \cos^4 x\ dx = \int \sin x\ (\cos^4 x - 2\cos^6 x + \cos^8 x)\ dx$$

$$= \int \sin x\ \cos^4 x\ dx - 2\int \sin x\ \cos^6 x\ dx\ + \int \sin x\ \cos^8 x\ dx$$

$$= -\frac{\cos^5 x}{5} + 2\ \frac{\cos^7 x}{7} - \frac{\cos^9 x}{9} + C$$

21

$$\int \sin^2 \theta\ d\theta = \int \frac{1}{2}(1 - \cos 2\theta)\ d\theta = \frac{1}{2}\int d\theta - \int \cos 2\theta\ d\theta$$

$$= \frac{\theta}{2} - \frac{1}{2}\Big(\frac{\sin 2\theta}{2}\Big) + C = \frac{\theta}{2} - \frac{\sin 2\theta}{4} + C$$

23

(a) $\int x\ \cos x\ dx$ Integrate by parts, let $u = x$ and $dv = \cos x\ dx$

(b) $\int \cos x\ \sin^2 x\ dx$ Integrate by substitution, let $u = \sin x$

(c) $\int \frac{x}{x^2-4x-5}\,dx$ Integrate by partial fractions $\frac{A}{x+1}+\frac{B}{x-5}$

(d) $\int \frac{x-2}{x^2-4x+5}\,dx$ Integrate by substitution, let $u = x^2-4x+5$

(e) $\int \frac{\ln x}{x}\,dx$ Integrate by substitution, let $u=\ln x$

(f) $\int \ln x\,dx$ Integrate by parts, let $u=\ln x$ and $dv=dx$

25 $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \Rightarrow y=\pm b\sqrt{1-\frac{x^2}{a^2}}$ The area inside the first quadrant of the ellipse is $A = \int_0^a b\sqrt{1-\frac{x^2}{a^2}}\,dx$. Let $x=a\sin\theta \Rightarrow dx = a\cos\theta d\theta$. Thus,

$$\begin{aligned} A &= \int_0^{\frac{\pi}{2}} b\sqrt{1-\sin^2\theta}\cdot a\cos\theta\;d\theta \\ &= ab\int_0^{\frac{\pi}{2}}\cos^2\theta\;d\theta \\ &= ab[\frac{\theta}{2}-\frac{\sin 2\theta}{4}]_0^{\frac{\pi}{2}} \\ &= \frac{ab\pi}{4} \end{aligned}$$

The area of the ellipse is $4A = ab\pi$.

26 (a) For $p>1$,

$$\int_1^\infty x^{-p}\;dx = \frac{1}{1-p}x^{1-p}\Big|_1^\infty = \frac{1}{1-p}(0-1) = \frac{1}{p-1}.$$

Note that since $p>1$, $1-p<0$ and $\lim_{x\to\infty} x^{1-p}=0$.

(b) For $p=1$,

$$\int_1^\infty \frac{1}{x}\;dx = \ln x\Big|_1^\infty.$$

Since $\lim_{x\to\infty}\ln x = \infty$, the integral diverges.

(c) For $p<1$,

$$\int_1^\infty x^{-p}\;dx = \frac{1}{1-p}x^{1-p}\Big|_1^\infty.$$

In this case, $\lim_{x\to\infty} x^{1-p}=\infty$ since $1-p>0$. Therefore the integral diverges. (Or one simply notices that the integrant $\frac{1}{x^p}$ goes to ∞ as x goes to ∞, hence the integral diverges.)