

1. (a) A sequence is an ordered list of numbers. It can also be defined as a function whose domain is the set of positive integers.
(b) The terms a_n approach 8 as n becomes large. In fact, we can make a_n as close to 8 as we like by taking n sufficiently large.
(c) The terms a_n become large as n becomes large. In fact, we can make a_n as large as we like by taking n sufficiently large.

12. $a_n = \frac{\sqrt{n}}{1 + \sqrt{n}} = \frac{1}{1/\sqrt{n} + 1}$, so $a_n \rightarrow \frac{1}{0 + 1} = 1$ as $n \rightarrow \infty$. Converges

16. $a_n = \frac{(-1)^n n^3}{n^3 + 2n^2 + 1}$. Now $|a_n| = \frac{n^3}{n^3 + 2n^2 + 1} = \frac{1}{1 + \frac{2}{n} + \frac{1}{n^3}} \rightarrow 1$ as $n \rightarrow \infty$, but the terms of the sequence $\{a_n\}$ alternate in sign, so the sequence a_1, a_3, a_5, \dots converges to -1 and the sequence a_2, a_4, a_6, \dots converges to $+1$. This shows that the given sequence diverges since its terms don't approach a single real number.

18. $a_n = \cos(2/n)$. As $n \rightarrow \infty$, $2/n \rightarrow 0$, so $\cos(2/n) \rightarrow \cos 0 = 1$. Converges

35. (a) $a_n = 1000(1.06)^n \Rightarrow a_1 = 1060, a_2 = 1123.60, a_3 = 1191.02, a_4 = 1262.48, \text{ and } a_5 = 1338.23$.

(b) $\lim_{n \rightarrow \infty} a_n = 1000 \lim_{n \rightarrow \infty} (1.06)^n$, so the sequence diverges by (6) with $r = 1.06 > 1$.

2. $\sum_{n=1}^{\infty} a_n = 5$ means that by adding sufficiently many terms of the series we can get as close as we like to the number 5.

In other words, it means that $\lim_{n \rightarrow \infty} s_n = 5$, where s_n is the n th partial sum, that is, $\sum_{i=1}^n a_i$.

14. $\sum_{n=1}^{\infty} \frac{(-6)^{n-1}}{5^{n-1}}$ is a geometric series with $a = 1$ and $r = -\frac{6}{5}$. The series diverges since $|r| = \frac{6}{5} > 1$.

18. $\sum_{n=1}^{\infty} \frac{n+1}{2n-3}$ diverges since $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{n+1}{2n-3} = \frac{1}{2} \neq 0$. [Use (7), the Test for Divergence.]

20. $\sum_{k=1}^{\infty} \frac{k(k+2)}{(k+3)^2}$ diverges by (7), the Test for Divergence, since $\lim_{k \rightarrow \infty} a_k = \lim_{k \rightarrow \infty} \frac{k(k+2)}{(k+3)^2} = \lim_{k \rightarrow \infty} \frac{1 \cdot (1+2/k)}{(1+3/k)^2} = 1 \neq 0$.

26. $\sum_{k=1}^{\infty} (\cos 1)^k$ is a geometric series with ratio $r = \cos 1 \approx 0.540302$. It converges because $|r| < 1$. Its sum is

$$\frac{\cos 1}{1 - \cos 1} \approx 1.175343.$$

36. $\sum_{n=0}^{\infty} 2^n(x+1)^n = \sum_{n=0}^{\infty} [2(x+1)]^n = \sum_{n=1}^{\infty} [2(x+1)]^{n-1}$ is a geometric series with $r = 2(x+1)$, so the series

$$\text{converges} \Leftrightarrow |r| < 1 \Leftrightarrow |2(x+1)| < 1 \Leftrightarrow |x+1| < \frac{1}{2} \Leftrightarrow -\frac{1}{2} < x+1 < \frac{1}{2} \Leftrightarrow -\frac{3}{2} < x < -\frac{1}{2}.$$

In that case, the sum of the series is $\frac{a}{1-r} = \frac{1}{1-2(x+1)} = \frac{1}{-1-2x}$ or $\frac{-1}{2x+1}$.

43. (a) The first step in the chain occurs when the local government spends D dollars. The people who receive it spend a fraction c of those D dollars, that is, Dc dollars. Those who receive the Dc dollars spend a fraction c of it, that is, Dc^2 dollars. Continuing in this way, we see that the total spending after n transactions is

$$S_n = D + Dc + Dc^2 + \cdots + Dc^{n-1} = \frac{D(1-c^n)}{1-c} \text{ by (3).}$$

$$\begin{aligned} \text{(b) } \lim_{n \rightarrow \infty} S_n &= \lim_{n \rightarrow \infty} \frac{D(1-c^n)}{1-c} = \frac{D}{1-c} \lim_{n \rightarrow \infty} (1-c^n) = \frac{D}{1-c} \quad [\text{since } 0 < c < 1 \Rightarrow \lim_{n \rightarrow \infty} c^n = 0] \\ &= \frac{D}{s} \quad [\text{since } c+s=1] = kD \quad [\text{since } k=1/s] \end{aligned}$$

If $c = 0.8$, then $s = 1 - c = 0.2$ and the multiplier is $k = 1/s = 5$.

44. (a) Initially, the ball falls a distance H , then rebounds a distance rH , falls rH , rebounds r^2H , falls r^2H , etc. The total distance it travels is

$$\begin{aligned} H + 2rH + 2r^2H + 2r^3H + \cdots &= H(1 + 2r + 2r^2 + 2r^3 + \cdots) = H[1 + 2r(1 + r + r^2 + \cdots)] \\ &= H\left[1 + 2r\left(\frac{1}{1-r}\right)\right] = H\left(\frac{1+r}{1-r}\right) \text{ meters} \end{aligned}$$

- (b) From Example 3 in Section 2.1, we know that a ball falls $\frac{1}{2}gt^2$ meters in t seconds, where g is the gravitational acceleration. Thus, a ball falls h meters in $t = \sqrt{2h/g}$ seconds. The total travel time in seconds is

$$\begin{aligned} \sqrt{\frac{2H}{g}} + 2\sqrt{\frac{2H}{g}r} + 2\sqrt{\frac{2H}{g}r^2} + 2\sqrt{\frac{2H}{g}r^3} + \cdots &= \sqrt{\frac{2H}{g}} [1 + 2\sqrt{r} + 2\sqrt{r^2} + 2\sqrt{r^3} + \cdots] \\ &= \sqrt{\frac{2H}{g}} (1 + 2\sqrt{r}[1 + \sqrt{r} + \sqrt{r^2} + \cdots]) = \sqrt{\frac{2H}{g}} \left[1 + 2\sqrt{r}\left(\frac{1}{1-\sqrt{r}}\right)\right] = \sqrt{\frac{2H}{g}} \frac{1+\sqrt{r}}{1-\sqrt{r}} \end{aligned}$$

- (c) It will help to make a chart of the time for each descent and each rebound of the ball, together with the velocity just before and just after each bounce. Recall that the time in seconds needed to fall h meters is $\sqrt{2h/g}$. The ball hits the ground with velocity $-g\sqrt{2h/g} = -\sqrt{2hg}$ (taking the upward direction to be positive) and rebounds with velocity $kg\sqrt{2h/g} = k\sqrt{2hg}$, taking time $k\sqrt{2h/g}$ to reach the top of its bounce, where its velocity is 0. At that point, its height

is $k^2 h$. All these results follow from the formulas for vertical motion with gravitational acceleration $-g$:

$$\frac{d^2 y}{dt^2} = -g \Rightarrow v = \frac{dy}{dt} = v_0 - gt \Rightarrow y = y_0 + v_0 t - \frac{1}{2}gt^2.$$

number of descent	time of descent	speed before bounce	speed after bounce	time of ascent	peak height
1	$\sqrt{2H/g}$	$\sqrt{2Hg}$	$k\sqrt{2Hg}$	$k\sqrt{2H/g}$	$k^2 H$
2	$\sqrt{2k^2 H/g}$	$\sqrt{2k^2 Hg}$	$k\sqrt{2k^2 Hg}$	$k\sqrt{2k^2 H/g}$	$k^4 H$
3	$\sqrt{2k^4 H/g}$	$\sqrt{2k^4 Hg}$	$k\sqrt{2k^4 Hg}$	$k\sqrt{2k^4 H/g}$	$k^6 H$
...

The total travel time in seconds is

$$\begin{aligned} \sqrt{\frac{2H}{g}} + k\sqrt{\frac{2H}{g}} + k\sqrt{\frac{2H}{g}} + k^2\sqrt{\frac{2H}{g}} + k^2\sqrt{\frac{2H}{g}} + \dots &= \sqrt{\frac{2H}{g}}(1 + 2k + 2k^2 + 2k^3 + \dots) \\ &= \sqrt{\frac{2H}{g}}[1 + 2k(1 + k + k^2 + \dots)] = \sqrt{\frac{2H}{g}}\left[1 + 2k\left(\frac{1}{1-k}\right)\right] = \sqrt{\frac{2H}{g}}\frac{1+k}{1-k} \end{aligned}$$

Another method: We could use part (b). At the top of the bounce, the height is $k^2 h = rh$, so $\sqrt{r} = k$ and the result follows from part (b).

50. If $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n \rightarrow \infty} a_n = 0$ by Theorem 6, so $\lim_{n \rightarrow \infty} \frac{1}{a_n} \neq 0$, and so $\sum_{n=1}^{\infty} \frac{1}{a_n}$ is divergent by the Test for

Divergence.

Mathematics 1b - Solution Set for Series A

1. A sum is geometric if

$$\frac{a_n}{a_{n-1}} = r$$

where r is a constant.

- (a) $\sum_{k=1}^{70} \left(\frac{1}{k}\right)$, $\frac{a_n}{a_{n-1}} = \frac{\frac{1}{n}}{\frac{1}{n-1}} = \frac{n-1}{n} \neq \text{constant}$. So it is not a geometric sum.
- (b) $\sum_{k=1}^{50} \left(\frac{1}{k}\right)^2$, $\frac{a_n}{a_{n-1}} = \frac{\left(\frac{1}{n}\right)^2}{\left(\frac{1}{n-1}\right)^2} = \frac{(n-1)^2}{n^2} \neq \text{constant}$. So it is not a geometric sum.
- (c) $\sum_{k=1}^{60} \left(\frac{1}{k}\right)^k$, $\frac{a_n}{a_{n-1}} = \frac{\left(\frac{1}{n}\right)^n}{\left(\frac{1}{n-1}\right)^{n-1}} = \frac{(n-1)^{n-1}}{n^n} \neq \text{constant}$. So it is not a geometric sum.
- (d) $\sum_{k=1}^{60} (1.01)^{\frac{k}{3}}$, $\frac{a_n}{a_{n-1}} = \frac{(1.01)^{\frac{n}{3}}}{(1.01)^{\frac{n-1}{3}}} = \frac{(1.01)^{\frac{n}{3}}}{(1.01)^{\frac{n}{3}}(1.01)^{-\frac{1}{3}}} = (1.01)^{\frac{1}{3}} = r$ and we can calculate S_n which is, when $n = 60$,

$$S_n = \frac{a_0(1 - r^n)}{1 - r} = \frac{(1.01)^{\frac{1}{3}}}{1 - (1.01)^{\frac{1}{3}}} [1 - (1.01)^{20}]$$

2. These are all geometric sums with $r = \frac{1}{3}$.

- (a) $\sum_{k=0}^{100} \left(\frac{1}{3}\right)^k$, so $a_0 = 1$, $n = 101$ and $r = \frac{1}{3}$. Thus we have:

$$\sum_{k=0}^{100} \left(\frac{1}{3}\right)^k = \frac{1(1 - \left(\frac{1}{3}\right)^{101})}{1 - \frac{1}{3}} = \frac{3}{2} \left(1 - \left(\frac{1}{3}\right)^{101}\right)$$

- (b) $\sum_{k=0}^{\infty} \left(\frac{1}{3}\right)^k = \frac{1}{1 - \frac{1}{3}} = \frac{3}{2}$

- (c) We have $a_0 = \frac{1}{9}$, $n = 99$ and $r = \frac{1}{3}$. Thus,

$$\sum_{k=2}^{100} \left(\frac{1}{3}\right)^k = \frac{\frac{1}{9}(1 - \left(\frac{1}{3}\right)^{99})}{1 - \frac{1}{3}} = \frac{1}{6} \left(1 - \left(\frac{1}{3}\right)^{99}\right)$$

- (d) $\sum_{k=2}^{\infty} \left(\frac{1}{3}\right)^k = \frac{\frac{1}{9}}{1 - \frac{1}{3}} = \frac{1}{6}$

3. (a) No. If you have a series that converges and you add a finite number of terms to the beginning of the series, the series still converges. Hence, the first few terms do not affect whether the series converges.
- (b) Yes. If you have a convergent series whose sum is S , then if you add a term to the beginning of the series whose value is a , then the sum of the series is $S + a$.

4. (a) Infinite geometric series, so converges if $|x| < 1$ and diverges if $|x| \geq 1$. Because if $|x| \geq 1$ then $\lim_{n \rightarrow \infty} x^n \not\rightarrow 0$ so the series could not converge. For $|x| < 1$, we have

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$$

- (b) $\sum_{k=0}^{\infty} (x-4)^k$, is again a geometric series with $r = x-4$. The series diverges if $|x-4| \geq 1$ if $x \geq 5$ or $x \leq 3$. The series converges if $|x-4| < 1$, i.e. if $3 < x < 5$. And the sum is

$$\sum_{k=0}^{\infty} (x-4)^k = \frac{1}{1-(x-4)} = \frac{1}{5-x}$$

7. $f(x) = \sin(\pi x)$ would change her mind. Because for any integer n , $\sin(n\pi) = 0$ so $\lim_{n \rightarrow \infty} \sin(n\pi) = 0$, but for x being a real number the $\lim_{n \rightarrow \infty} \sin(x\pi)$ doesn't exist because the sine function oscillates between -1 and 1.

Note that for (a), Neither $\sin n$, $n \rightarrow \infty$ nor $\sin x$, $x \rightarrow \infty$ exists. For (b), both $\frac{\sin n}{n}$, $n \rightarrow \infty$ and $\frac{\sin x}{x}$, $x \rightarrow \infty$ converge to 0.