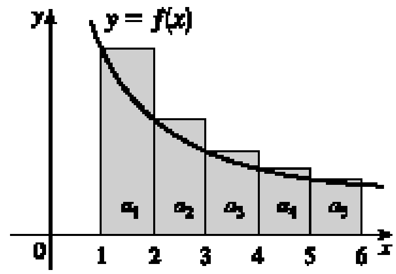
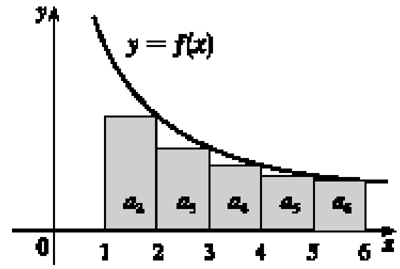
2. From the first figure, we see that $\int_1^6 f(x)dx < \sum_{i=1}^{5} a_i$. From the second figure, we see that $\sum_{i=2}^{6} a_i < \int_1^6 f(x)\,dx$. Thus, we have $\sum_{i=2}^{6} a_i < \int_1^6 f(x)\,dx < \sum_{i=1}^{5} a_i$.





5. $\sum_{n=1}^{\infty} n^b$ is a p-series with $p = -b$. $\sum_{n=1}^{\infty} b^n$ is a geometric series. By (1), the p-series is convergent if $p > 1$. In this case, $\sum_{n=1}^{\infty} n^b = \sum_{n=1}^{\infty} (1/n^{-b})$, so $-b > 1 \quad \Leftrightarrow \quad b < -1$ are the values for which the series converge. A geometric series $\sum_{n=1}^{\infty} ar^{n-1}$ converges if $|r| < 1$, so $\sum_{n=1}^{\infty} b^n$ converges if $|b| < 1 \quad \Leftrightarrow \quad -1 < b < 1$.

14. $f(x) = \dfrac{x^2}{x^3+1}$ is continuous and positive on $[2, \infty)$, and also decreasing since $f'(x) = \dfrac{x(2-x^3)}{(x^3+1)^2} < 0$ for $x \geq 2$, so we can use the Integral Test [note that f is *not* decreasing on $[1, \infty)$].

$\displaystyle\int_2^{\infty} \frac{x^2}{x^3+1}\,dx = \lim_{t\to\infty} \left[\tfrac{1}{3}\ln(x^3+1)\right]_2^t = \tfrac{1}{3}\lim_{t\to\infty}\left[\ln(t^3+1) - \ln 9\right] = \infty$, so the series $\displaystyle\sum_{n=2}^{\infty} \frac{n^2}{n^3+1}$ diverges, and so does the given series, $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{n^3+1}$.

Another solution: Use the Limit Comparison Test with $a_n = \dfrac{n^2}{n^3+1}$ and $b_n = \dfrac{1}{n}$:

$\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^2 \cdot n}{n^3+1} = \lim_{n\to\infty} \frac{1}{1+1/n^3} = 1 > 0$. Since the harmonic series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, so does $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{n^3+1}$.

20. $\dfrac{1}{\sqrt{n^3+1}} < \dfrac{1}{\sqrt{n^3}} = \dfrac{1}{n^{3/2}}$, so $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3+1}}$ converges by comparison with the convergent p-series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ $\left(p = \tfrac{3}{2} > 1\right)$.

24. $\dfrac{1+\sin n}{10^n} \leq \dfrac{2}{10^n}$ and $\displaystyle\sum_{n=0}^{\infty} \frac{2}{10^n} = 2\sum_{n=0}^{\infty}\left(\frac{1}{10}\right)^n$, so the given series converges by comparison with a constant multiple of a convergent geometric series.

26. If $a_n = \dfrac{n+5}{\sqrt[3]{n^7+n^2}}$ and $b_n = \dfrac{n}{\sqrt[3]{n^7}} = \dfrac{n}{n^{7/3}} = \dfrac{1}{n^{4/3}}$, then

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^{7/3}+5n^{4/3}}{(n^7+n^2)^{1/3}} \cdot \frac{n^{-7/3}}{n^{-7/3}} = \lim_{n\to\infty} \frac{1+5/n}{\left[(n^7+n^2)/n^7\right]^{1/3}} = \lim_{n\to\infty} \frac{1+5/n}{(1+1/n^5)^{1/3}} = \frac{1+0}{(1+0)^{1/3}} = 1 > 0,$$

so $\sum\limits_{n=1}^{\infty} \dfrac{n+5}{\sqrt[3]{n^7+n^2}}$ converges by the Limit Comparison Test with the convergent p-series $\sum\limits_{n=1}^{\infty} \dfrac{1}{n^{4/3}}$.

37. Since $\dfrac{d_n}{10^n} \le \dfrac{9}{10^n}$ for each n, and since $\sum\limits_{n=1}^{\infty} \dfrac{9}{10^n}$ is a convergent geometric series $\left(|r| = \frac{1}{10} < 1\right)$, $0.d_1d_2d_3\ldots = \sum\limits_{n=1}^{\infty} \dfrac{d_n}{10^n}$ will always converge by the Comparison Test.

39. Yes. Since $\sum a_n$ is a convergent series with positive terms, $\lim\limits_{n\to\infty} a_n = 0$ by Theorem 8.2.6, and $\sum b_n = \sum \sin(a_n)$ is a series with positive terms (for large enough n). We have $\lim\limits_{n\to\infty} \dfrac{b_n}{a_n} = \lim\limits_{n\to\infty} \dfrac{\sin(a_n)}{a_n} = 1 > 0$ by Theorem 3.4.2. Thus, $\sum b_n$ is also convergent by the Limit Comparison Test.

8. $\sum\limits_{n=1}^{\infty} (-1)^{n-1}\left(\dfrac{\ln n}{n}\right) = 0 + \sum\limits_{n=2}^{\infty} (-1)^{n-1}\left(\dfrac{\ln n}{n}\right)$. $b_n = \dfrac{\ln n}{n} > 0$ for $n \ge 2$, and if $f(x) = \dfrac{\ln x}{x}$, then

$f'(x) = \dfrac{1-\ln x}{x^2} < 0$ for $x > e$, so $\{b_n\}$ is eventually decreasing. Also,

$\lim\limits_{n\to\infty} b_n = \lim\limits_{n\to\infty} \dfrac{\ln n}{n} = \lim\limits_{x\to\infty} \dfrac{\ln x}{x} \overset{\text{H}}{=} \lim\limits_{x\to\infty} \dfrac{1/x}{1} = 0$, so the series converges by the Alternating Series Test.

12. The series $\sum\limits_{n=1}^{\infty} \dfrac{(-1)^n}{n\,5^n}$ satisfies (i) of the Alternating Series Test because $\dfrac{1}{(n+1)5^{n+1}} < \dfrac{1}{n\,5^n}$ and (ii) $\lim\limits_{n\to\infty} \dfrac{1}{n\,5^n} = 0$, so the series is convergent. Now $b_4 = \dfrac{1}{4\cdot 5^4} = 0.0004 > 0.0001$ and $b_5 = \dfrac{1}{5\cdot 5^5} = 0.000064 < 0.0001$, so by the Alternating Series Estimation Theorem, $n = 4$. (That is, since the 5th term is less than the desired error, we need to add the first 4 terms to get the sum to the desired accuracy.)

14. Using the Ratio Test with the series $\sum_{n=1}^{\infty}(-1)^{n-1}ne^{-n} = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{n}{e^n}$,

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^n(n+1)}{e^{n+1}}\cdot\frac{e^n}{(-1)^{n-1}n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^1(n+1)}{en}\right| = \frac{1}{e}\lim_{n\to\infty}\frac{n+1}{n} = \frac{1}{e}(1) = \frac{1}{e} < 1,$$

so the series is absolutely convergent (and therefore convergent). Now $b_6 = 6/e^6 \approx 0.015 > 0.01$ and $b_7 = 7/e^7 \approx 0.006 < 0.01$, so by the Alternating Series Estimation Theorem, $n = 6$. (That is, since the 7th term is less than the desired error, we need to add the first 6 terms to get the sum to the desired accuracy.)

20. The series $\sum_{n=1}^{\infty}\frac{n^2}{2^n}$ has positive terms and $\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\left[\frac{(n+1)^2}{2^{n+1}}\cdot\frac{2^n}{n^2}\right] = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^2\cdot\frac{1}{2} = \frac{1}{2} < 1$, so the series is absolutely convergent by the Ratio Test.

24. $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{2^n}{n^4}$ diverges by the Test for Divergence. $\lim_{n\to\infty}\frac{2^n}{n^4} = \infty$, so $\lim_{n\to\infty}(-1)^{n-1}\frac{2^n}{n^4}$ does not exist.

33. (a) $\lim_{n\to\infty}\left|\frac{1/(n+1)^3}{1/n^3}\right| = \lim_{n\to\infty}\frac{n^3}{(n+1)^3} = \lim_{n\to\infty}\frac{1}{(1+1/n)^3} = 1.$ Inconclusive

(b) $\lim_{n\to\infty}\left|\frac{(n+1)}{2^{n+1}}\cdot\frac{2^n}{n}\right| = \lim_{n\to\infty}\frac{n+1}{2n} = \lim_{n\to\infty}\left(\frac{1}{2}+\frac{1}{2n}\right) = \frac{1}{2}.$ Conclusive (convergent)

(c) $\lim_{n\to\infty}\left|\frac{(-3)^n}{\sqrt{n+1}}\cdot\frac{\sqrt{n}}{(-3)^{n-1}}\right| = 3\lim_{n\to\infty}\sqrt{\frac{n}{n+1}} = 3\lim_{n\to\infty}\sqrt{\frac{1}{1+1/n}} = 3.$ Conclusive (divergent)

(d) $\lim_{n\to\infty}\left|\frac{\sqrt{n+1}}{1+(n+1)^2}\cdot\frac{1+n^2}{\sqrt{n}}\right| = \lim_{n\to\infty}\left[\sqrt{1+\frac{1}{n}}\cdot\frac{1/n^2+1}{1/n^2+(1+1/n)^2}\right] = 1.$ Inconclusive

Series Handout

9. (a) *By direct comparison.* Note that $\frac{1}{n+2} > \frac{1}{n+n}$ for $n > 2$, so $\frac{1}{n+2} > \frac{1}{2n}$ for $n > 2$. Thus

$$\sum_{n=3}^{\infty} \frac{1}{n+2} > \sum_{n=3}^{\infty} \frac{1}{2n} = \frac{1}{2} \sum_{n=3}^{\infty} \frac{1}{n}.$$

But $\sum \frac{1}{n}$ diverges because it is the harmonic series and $\sum \frac{1}{n+2}$ has infinitely many terms greater than the harmonic series so it diverges as well.

By limit comparison. The sequence $\frac{1}{n+2}$ "looks like" $\frac{1}{n}$ and we know that $\sum_{n=1}^{\infty} 1/n$ diverges. Consider the limit of the ratio of these two sequences:

$$\lim_{n\to\infty} \frac{\frac{1}{n+2}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{n+2} = 1.$$

So the series $\sum_1 \infty$ diverges as well.

By integral test.

$$\int_1^{\infty} \frac{1}{x+2}\, dx = (\ln|x+2|)\Big|_1^{\infty}$$

diverges. Thus the series diverges as well.

(b) *By direct comparison.* Note that

$$\frac{1}{\sqrt{n^2+10}} > \frac{1}{\sqrt{n^2+n^2}} \qquad \text{for } n > 3.$$

But

$$\frac{1}{\sqrt{n^2+n^2}} = \frac{1}{\sqrt{2n^2}} = \frac{1}{n\sqrt{2}}.$$

So again

$$\sum_{n=4}^{\infty} \frac{1}{\sqrt{n^2+10}} > \sum_{n=4}^{\infty} \frac{1}{n\sqrt{2}} = \frac{1}{\sqrt{2}} \sum_{n=4}^{\infty} \frac{1}{n}$$

which diverges. So

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+10}}$$

must also diverge by the same logic as above.

By limit comparison. The sequence $\frac{1}{\sqrt{n^2+10}}$ "looks like" $\frac{1}{\sqrt{n^2}} = \frac{1}{n}$ and $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges. Consider the limit of the ratio:

$$\lim_{n\to\infty} \frac{\frac{1}{\sqrt{n^2+10}}}{\frac{1}{n}} = \lim_{n\to\infty} \sqrt{\frac{n^2}{n^2+10}} = 1.$$

Thus the series $\sum_{n=1}^{\infty} 1/\sqrt{n^2+10}$ diverges as well.

(c) *By integral test.* Let $u = \ln x$. Then $du = \frac{1}{x} dx$.

$$\int_2^{\infty} \frac{1}{x \ln x} = \int_{\ln 2}^{\infty} \frac{1}{u}\, du = (\ln|u|)\Big|_{\ln 2}^{\infty}$$

diverges to infinity. Thus the series $\sum_{n=2}^{\infty} 1/(n \ln n)$ diverges as well.

(d) *By integral test.* Let $u = \ln x$. Then $du = \frac{1}{x} dx$.

$$\int_2^\infty \frac{1}{x(\ln x)^p} = \int_{\ln 2}^\infty \frac{1}{u^p}\ du$$

which converges as long as $p > 1$. Thus the series $\sum_{n=2}^\infty 1/(n(\ln n)^p)$ converges as well if $p > 1$.

13. This is an alternating series. Look for the first a_{n+1} such that

$$|a_{n+1}| = \left| \frac{(-0.1)^{n+1}}{(n+1)!} \right| < 10^{-8}.$$

Through trial and error, $n = 5$. You need the first six terms. Don't forget to include the $n = 0$ term.

14. Same idea as above.

$$|a_n| = \left| \frac{(-1)^n (0.2)^{2n+1}}{(2n+1)!} \right| < 10^{-6}.$$

Through trial and error, $n = 2$ works. So the first three non-zero terms. Don't forget to include the $n = 0$ term. This approximation is too big, because it ended on a positive term and is an alternating series.

15. (a) DIVERGES since

$$\lim_{k\to\infty} \frac{2k^2 - 10k}{10k^2 + 5k} = \frac{1}{5} \neq 0.$$

It violates condition (iii): terms do not go to zero.

(b) CONVERGES. Note that $|\sin k| \leq 1$.

$$\sum_{k=1}^\infty \left| (-1)^k \frac{\sin k}{\sqrt{k^3}} \right| \leq \sum_{k=1}^\infty \frac{1}{\sqrt{k^3}}$$

which converges since it is a p-series for $p = 3/2 > 1$. Thus the original series converges (absolutely) as well by absolute convergence test. It violates condition (ii): magnitude of terms is not decreasing and also violates condition (i): it is not really alternating!

18. $\sum_{n=0}^\infty \frac{(x-3)^n}{5^n}$ is geometric with ratio $r = \frac{(x-3)}{5}$. So the series would converge when $|r| < 1 \Rightarrow -1 < \frac{x-3}{5} < 1 \Rightarrow -5 < x - 3 < 5 \Rightarrow -2 < x < 8$. Now, applying the ratio test, we get:

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{\frac{(x-3)^{n+1}}{5^{n+1}}}{\frac{(x-3)^n}{5^n}} \right| = \lim_{n\to\infty} \left| \frac{x-3}{5} \right| = \left| \frac{x-3}{5} \right|.$$

It converges when

$$\left| \frac{x-3}{5} \right| < 1,$$

exactly what we got by applying what we know about geometric series.