Name: ___________________ Class:___ Date: _____________

9. $f(x) = \dfrac{x}{9+x^2} = \dfrac{x}{9}\left[\dfrac{1}{1+(x/3)^2}\right] = \dfrac{x}{9}\left[\dfrac{1}{1-\{-(x/3)^2\}}\right] = \dfrac{x}{9}\sum_{n=0}^{\infty}\left[-\left(\dfrac{x}{3}\right)^2\right]^n = \dfrac{x}{9}\sum_{n=0}^{\infty}(-1)^n\dfrac{x^{2n}}{9^n}$

$= \sum_{n=0}^{\infty}(-1)^n\dfrac{x^{2n+1}}{9^{n+1}}$

The geometric series $\sum_{n=0}^{\infty}\left[-\left(\dfrac{x}{3}\right)^2\right]^n$ converges when $\left|-\left(\dfrac{x}{3}\right)^2\right| < 1 \quad\Leftrightarrow\quad \dfrac{|x^2|}{9} < 1 \quad\Leftrightarrow$

$|x|^2 < 9 \quad\Leftrightarrow\quad |x| < 3$, so $R = 3$ and $I = (-3, 3)$.

12. (a) $\dfrac{1}{1+x} = \dfrac{1}{1-(-x)} = \sum_{n=0}^{\infty}(-1)^n x^n$ [geometric series with $R = 1$], so

$f(x) = \ln(1+x) = \int \dfrac{dx}{1+x} = \int\left[\sum_{n=0}^{\infty}(-1)^n x^n\right]dx = C + \sum_{n=0}^{\infty}(-1)^n\dfrac{x^{n+1}}{n+1} = \sum_{n=1}^{\infty}\dfrac{(-1)^{n-1}x^n}{n}$

[$C = 0$ since $f(0) = \ln 1 = 0$], with $R = 1$

(b) $f(x) = x\ln(1+x) = x\left[\sum_{n=1}^{\infty}\dfrac{(-1)^{n-1}x^n}{n}\right]$ [by part (a)] $= \sum_{n=1}^{\infty}\dfrac{(-1)^{n-1}x^{n+1}}{n} = \sum_{n=2}^{\infty}\dfrac{(-1)^n x^n}{n-1}$ with $R = 1$.

(c) $f(x) = \ln(x^2+1) = \sum_{n=1}^{\infty}\dfrac{(-1)^{n-1}(x^2)^n}{n}$ [by part (a)] $= \sum_{n=1}^{\infty}\dfrac{(-1)^{n-1}x^{2n}}{n}$ with $R = 1$.

28. $\int_0^{0.3}\dfrac{x^2}{1+x^4}\,dx = \int_0^{0.3}x^2\sum_{n=0}^{\infty}(-1)^n x^{4n}\,dx = \sum_{n=0}^{\infty}\left[\dfrac{(-1)^n x^{4n+3}}{4n+3}\right]_0^{0.3} = \sum_{n=0}^{\infty}\dfrac{(-1)^n\,3^{4n+3}}{(4n+3)10^{4n+3}}$

$= \dfrac{3^3}{3\times 10^3} - \dfrac{3^7}{7\times 10^7} + \dfrac{3^{11}}{11\times 10^{11}} - \cdots$

The series is alternating, so if we use only two terms, the error is at most $\dfrac{3^{11}}{11\times 10^{11}} \approx 0.000\,000\,16$. So, to six decimal places, $\int_0^{0.3}\dfrac{x^2}{1+x^4}\,dx \approx \dfrac{3^3}{3\times 10^3} - \dfrac{3^7}{7\times 10^7} \approx 0.008\,969$.

2. (a) Using Formula 6, a power series expansion of f at 1 must have the form $f(1) + f'(1)(x-1) + \cdots$. Comparing to the given series, $1.6 - 0.8(x-1) + \cdots$, we must have $f'(1) = -0.8$. But from the graph, $f'(1)$ is positive. Hence, the given series is *not* the Taylor series of f centered at 1.

(b) A power series expansion of f at 2 must have the form $f(2) + f'(2)(x-2) + \frac{1}{2}f''(2)(x-2)^2 + \cdots$. Comparing to the given series, $2.8 + 0.5(x-2) + 1.5(x-2)^2 - 0.1(x-2)^3 + \cdots$, we must have $\frac{1}{2}f''(2) = 1.5$; that is, $f''(2)$ is positive. But from the graph, f is concave downward near $x = 2$, so $f''(2)$ must be negative. Hence, the given series is *not* the Taylor series of f centered at 2.

16.

n	$f^{(n)}(x)$	$f^{(n)}(1)$
0	x^{-2}	1
1	$-2x^{-3}$	-2
2	$6x^{-4}$	6
3	$-24x^{-5}$	-24
4	$120x^{-6}$	120
$\vdots$	$\vdots$	$\vdots$

$$x^{-2} = 1 - 2(x-1) + 6 \cdot \frac{(x-1)^2}{2!} - 24 \cdot \frac{(x-1)^3}{3!} + 120 \cdot \frac{(x-1)^4}{4!} - \cdots$$
$$= 1 - 2(x-1) + 3(x-1)^2 - 4(x-1)^3 + 5(x-1)^4 - \cdots$$
$$= \sum_{n=0}^{\infty} (-1)^n (n+1)(x-1)^n.$$

$\lim\limits_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty} \frac{(n+2)\,|x-1|^{n+1}}{(n+1)\,|x-1|^n} = \lim\limits_{n\to\infty} \left[\frac{n+2}{n+1} \cdot |x-1|\right] = |x-1| < 1$ for convergence, so $R = 1$.

18. If $f(x) = \sin x$, then $f^{(n+1)}(x) = \pm \sin x$ or $\pm \cos x$. In each case, $\left|f^{(n+1)}(x)\right| \le 1$, so by Formula 9 with $a = 0$ and $M = 1$, $|R_n(x)| \le \frac{1}{(n+1)!}\left|x - \frac{\pi}{2}\right|^{n+1}$. Thus, $|R_n(x)| \to 0$ as $n \to \infty$ by Equation 10. So $\lim\limits_{n\to\infty} R_n(x) = 0$ and, by Theorem 8, the series in Exercise 14 represents $\sin x$ for all x.

24. $\cos x = \sum\limits_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \quad\Rightarrow\quad \cos 2x = \sum\limits_{n=0}^{\infty} (-1)^n \frac{(2x)^{2n}}{(2n)!} = \sum\limits_{n=0}^{\infty} \frac{(-1)^n 2^{2n}}{(2n)!} x^{2n} \quad\Rightarrow$

$f(x) = x \cos 2x = \sum\limits_{n=0}^{\infty} \frac{(-1)^n 2^{2n}}{(2n)!} x^{2n+1}, \quad R = \infty$

34. $\frac{\sin x}{x} = \frac{1}{x} \sum\limits_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = \sum\limits_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n+1)!}$, so $\int \frac{\sin x}{x}\,dx = \int \sum\limits_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n+1)!}\,dx = C + \sum\limits_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)(2n+1)!}$

37. By Exercise 33, $\int x \cos(x^3)\,dx = C + \sum\limits_{n=0}^{\infty} (-1)^n \frac{x^{6n+2}}{(6n+2)(2n)!}$, so

$\int_0^1 x \cos(x^3)\,dx = \left[\sum\limits_{n=0}^{\infty} (-1)^n \frac{x^{6n+2}}{(6n+2)(2n)!}\right]_0^1 = \sum\limits_{n=0}^{\infty} \frac{(-1)^n}{(6n+2)(2n)!} = \frac{1}{2} - \frac{1}{8 \cdot 2!} + \frac{1}{14 \cdot 4!} - \frac{1}{20 \cdot 6!} + \cdots$, but $\frac{1}{20 \cdot 6!} = \frac{1}{14{,}400} \approx 0.000\,069$, so $\int_0^1 x \cos(x^3)\,dx \approx \frac{1}{2} - \frac{1}{16} + \frac{1}{336} \approx 0.440$ (correct to three decimal places) by the Alternating Series Estimation Theorem.

42. $$\lim_{x\to 0}\frac{1-\cos x}{1+x-e^x}=\lim_{x\to 0}\frac{1-\left(1-\frac{1}{2!}x^2+\frac{1}{4!}x^4-\frac{1}{6!}x^6+\cdots\right)}{1+x-\left(1+x+\frac{1}{2!}x^2+\frac{1}{3!}x^3+\frac{1}{4!}x^4+\frac{1}{5!}x^5+\frac{1}{6!}x^6+\cdots\right)}$$
$$=\lim_{x\to 0}\frac{\frac{1}{2!}x^2-\frac{1}{4!}x^4+\frac{1}{6!}x^6-\cdots}{-\frac{1}{2!}x^2-\frac{1}{3!}x^3-\frac{1}{4!}x^4-\frac{1}{5!}x^5-\frac{1}{6!}x^6-\cdots}$$
$$=\lim_{x\to 0}\frac{\frac{1}{2!}-\frac{1}{4!}x^2+\frac{1}{6!}x^4-\cdots}{-\frac{1}{2!}-\frac{1}{3!}x-\frac{1}{4!}x^2-\frac{1}{5!}x^3-\frac{1}{6!}x^4-\cdots}=\frac{\frac{1}{2}-0}{-\frac{1}{2}-0}=-1$$

since power series are continuous functions.

49. $\displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{x^{4n}}{n!}=\sum_{n=0}^{\infty}\frac{\left(-x^4\right)^n}{n!}=e^{-x^4}$, by (11).

52. $\displaystyle\sum_{n=0}^{\infty}\frac{3^n}{5^n\,n!}=\sum_{n=0}^{\infty}\frac{(3/5)^n}{n!}=e^{3/5}$, by (11).

4. $$(1-x)^{2/3}=\sum_{n=0}^{\infty}\binom{\frac{2}{3}}{n}(-x)^n$$
$$=1+\tfrac{2}{3}(-x)+\frac{\frac{2}{3}\left(-\frac{1}{3}\right)}{2!}(-x)^2+\frac{\frac{2}{3}\left(-\frac{1}{3}\right)\left(-\frac{4}{3}\right)}{3!}(-x)^3+\cdots$$
$$=1-\tfrac{2}{3}x+\sum_{n=2}^{\infty}\frac{(-1)^{n-1}(-1)^n\cdot 2\cdot[1\cdot 4\cdot 7\cdot\cdots\cdot(3n-5)]}{3^n\cdot n!}x^n$$
$$=1-\tfrac{2}{3}x-2\sum_{n=2}^{\infty}\frac{1\cdot 4\cdot 7\cdot\cdots\cdot(3n-5)}{3^n\cdot n!}x^n$$

and $|-x|<1 \quad\Leftrightarrow\quad |x|<1$, so $R=1$.

6. $$\frac{x^2}{\sqrt{2+x}}=\frac{x^2}{\sqrt{2\,(1+x/2)}}=\frac{x^2}{\sqrt{2}}\left(1+\frac{x}{2}\right)^{-1/2}=\frac{x^2}{\sqrt{2}}\sum_{n=0}^{\infty}\binom{-\frac{1}{2}}{n}\left(\frac{x}{2}\right)^n$$
$$=\frac{x^2}{\sqrt{2}}\left[1+\left(-\tfrac{1}{2}\right)\left(\frac{x}{2}\right)+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}\left(\frac{x}{2}\right)^2+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{3!}\left(\frac{x}{2}\right)^3+\cdots\right]$$
$$=\frac{x^2}{\sqrt{2}}+\frac{x^2}{\sqrt{2}}\sum_{n=1}^{\infty}(-1)^n\frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)}{n!\,2^{2n}}x^n$$
$$=\frac{x^2}{\sqrt{2}}+\sum_{n=1}^{\infty}(-1)^n\frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)}{n!\,2^{2n+1/2}}x^{n+2}\ \text{ and }\ \left|\frac{x}{2}\right|<1\quad\Leftrightarrow\quad |x|<2,\quad\text{so } R=2.$$

24. $\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots$. By the Alternating Series Estimation Theorem, the error is less than $\left|-\frac{1}{6!}x^6\right| < 0.005 \quad \Leftrightarrow$ $x^6 < 720(0.005) \quad \Leftrightarrow \quad |x| < (3.6)^{1/6} \approx 1.238$. The curves $y = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4$ and $y = \cos x + 0.005$ intersect at $x \approx 1.244$, so the graph confirms our estimate. Since both the cosine function and the given approximation are even functions, we need to check the estimate only for $x > 0$. Thus, the desired range of values for x is $-1.238 < x < 1.238$.