Answers to the First Math 1b Exam: October 27, 2004

1. (a) Slice the splat into concentric rings. The area of each ring will be approximately $2\pi x\Delta x$, and therefore the amount of mud for a single ring will be approximately $2\pi x e^{-x}\Delta x$. We can then write a Riemann sum

$$\text{Amount of mud} \approx 2\pi \sum_{i=1}^{n} x_i e^{-x_i}\Delta x.$$

(b) When we take the limit as n grows without bound we get

$$\text{Amount of mud} = \lim_{n\to\infty} 2\pi \sum_{i=1}^{n} x_i e^{-x_i}\Delta x = 2\pi \int_0^{12} x e^{-x} dx.$$

(c) Integrate by parts with $u = x$ and $dv = e^{-x}dx$ to get

$$\text{Amount of mud} = 2\pi(1 - 13e^{-12})$$

2. (a) The mountain must be sliced horizontally. Then, the density is approximately constant on each slice.

(b) Let's assume we choose the y-axis as our coordinate axis, going through the center of the base and through the top of the mountain. Also assume that the base is located at $y = 0$ and the top at $y = 8500$. Picking a $y_i \in [0, 8500]$ we find that the the mass of a small slice around y_i is given approximately by $m(y_i) = V(y_i) * \rho(y_i)$. The volume $V(y_i) \approx (\Delta y) * \pi * r^2(y_i)$. The radius $r(y_i)$ is simply given by $r(y_i) = \frac{10}{17}(8500 - y_i)$. Therefore, we find for the mass of this slice $m(y_i) \approx (\frac{10}{17}(8500 - y_i))^2\rho(y_i)\Delta y$.

(c) The total mass can be approximated by $M \approx \sum_{i=1}^{n} m(y_i) = \sum_{i=1}^{n}(\frac{10}{17}(8500 - y_i))^2\rho(y_i)\Delta y$.

(d) Therefore, the integral measuring to total mass is given by $M = \int_0^{8500}(\frac{10}{17}(8500 - y))^2\rho(y)dy$

3. Part (i) Partition the interval [0,1] into n equal subintervals, each with width $\Delta x = \frac{1-0}{n}$. We use the shell method. The volume of a shell is $2\pi rh\Delta x$.

A typical shell would have height e^{x_i}, radius $x_i + 1$ (because we are revolving around the line $x = -1$), and thickness Δx, so the volume of a typical shell is $2\pi(x_i + 1)e^{x_i}\Delta x$.

So a Riemann sum is given by $\displaystyle\sum_{i=1}^{n} 2\pi(x_i + 1)e^{x_i}\Delta x$

Taking the limit as n goes to infinity, we find the volume is given by $\int_0^1 2\pi(x + 1)e^x\, dx$

Part (ii) Partition the interval [0,1] into n equal subintervals, each with width $\Delta x = \frac{1-0}{n}$. We use the washer method.The volume of a washer is $\pi(R^2 - r^2)\Delta x$. A typical washer has outer radius equals to $e^{x_i} + 2$ and inner radius equals to 2 (because we are revolving around the line $y = -2$). So the volume of a typical washer is given by $\pi((e^{x_i} + 2)^2 - 2^2)\Delta x$.

Therefore a Riemann sum would be given by $\displaystyle\sum_{i=1}^{n} \pi((e^{x_i} + 2)^2 - 2^2)\Delta x$. Taking the limit as n goes to infinity, we find the volume is given by $\int_0^1 \pi((e^x + 2)^2 - 2^2)\, dx$.

4. (4a) Assume the rope has reached a height of x meters from the ground. The mass of rope that remains hanging is then $(10 - x)m \times 1kg/m = (10 - x)kg$, thus the total mass being supported is then

$$M(x) = (10 - x)kg + 5kg = (15 - x)kg.$$

Now, to support this mass requires a force of $F(x) = M(x)g$, thus the work required to lift this mass through a small distance Δx is

$$\Delta W = F(x)\Delta x = M(x)g\Delta x = (15 - x)g\Delta x.$$

Adding up the work required over all such small distances, we conclude that the total work required is given by

$$\int_0^{10} dW = \int_0^{10} F(x)dx = \int_0^{10} (15 - x)g dx \quad Joules.$$

(4b) Consider a small slice of rope that sits at a height of x meters from the ground, and represent the height of this slice by Δx. The mass of this slice is then $\Delta M = \rho(x)\Delta x\ kg$. Now, to support this mass requires a force of $\Delta F = \Delta M g$, and the distance to the top for this slice is $(10 - x)m$, thus the work required to lift this slice to the top is

$$\Delta W = \Delta F(10 - x).$$

Adding up the work required to lift all such slices, we conclude that the total work required to lift just the rope is given by

$$\int_0^{10} dW = \int_0^{10} dF(x)(10 - x) = \int_0^{10} \rho(x)(10 - x)dx \quad Joules$$

Adding now, seperately, the work required to lift the granite, we get an answer of

$$5 \times g \times 10 + \int_0^{10} \rho(x)(10 - x)dx \quad Joules.$$

5. (a) We should slice in such a way that in each slice the density is approximately constant. Since the density depends only the distance from the pipe, it is the distance from the slice that should be approximately contant on each slice. So we want to slice parallel to the pipe i.e. the x-axis.

(b) The first thing to note is that the total harvest is twice the harvest on the $y > 0$ part of the field. Slice this part of field as explained before that is, slice the interval $[01000\sqrt{2}/2]$ ito n equal portions with coordinates $y_1, \cdots, y_n$. Each slice is approximately a rectangle with width Δy and length $h(y_i)$. To get an expression for $h(y)$ you can for example use similar triangles:

$$\frac{h(y)}{1000\sqrt{2}/2 - y} = \frac{1000\sqrt{2}/2}{1000\sqrt{2}/2} = 1. \tag{1}$$

Since the harvest density is approximately contant on a slice, the harvest on it is approximately area*density there i.e. $h(y_i) * \Delta y * H(y_i)$. So we get approximation for the total harvest:

$$2\sum_{i=1}^{n}(1000\sqrt{2}/2 - y_i) * \Delta y * H(y_i). \tag{2}$$

As n tends to infinity this tends to

$$2\int_0^{1000\sqrt{2}/2}(1000\sqrt{2}/2 - y)H(y)dy \tag{3}$$

and this is the final answer.

6. (a) Use substitution. $u = \ln x$.

$$\int_e^T \frac{1}{x\ln x}dx = \int_{\ln e}^{\ln T}\frac{1}{u}du = \ln\ln T.$$

(b) Since $\lim_{T\to\infty}\ln\ln T = \infty$. The improper integral diverges.

(c) $\int_e^\infty 1/x dx$ diverges.

Using the comparison alone, we can not determine whether or not $\int_e^\infty \frac{1}{x\ln x}dx$ converges. because the divergence of the integral of a bigger function does not imply the divergence of that of the smaller function.

(d) Since

$$\sqrt{\frac{1}{x\ln x}} \geq \frac{1}{x\ln x},$$

for $x > e$, a comparsion argument shows that $\int_e^\infty \sqrt{\frac{1}{x\ln x}}dx$ diverges.

7. The graph $y = x^2$ is also the graph $x = \sqrt{y}$. The condition that c divides the given region into two regions of equal area can be written as

$$\int_0^c \sqrt{y} = \int_c^9 \sqrt{y}. \tag{4}$$

Since $\int \sqrt{y} = 2/3y^{3/2}$ this in turn transaltes into:

$$2/3c^{3/2} = 2/3 * 9^{3/2} - 2/3C^{3/2} \tag{5}$$

i.e.

$$2c^{3/2} = 9^{3/2}. \tag{6}$$

Therefore we get $c = 9/4^{1/3}$.