Review of Series

Math 1b

May 13, 2005

Tests for Convergence

Strategies for Convergence Tests

Take all lists with a grain of salt.

1. If the series is of the form $\sum \frac{1}{n^p}$, it is a p-series, which we know to be convergent if $p > 1$ and divergent if $p \leq 1$.

Example. Determine the convergence or divergence of the following series.

(i) $$\sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$$

(ii) $$\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$$

(iii) $$\sum_{n=1}^{\infty} \frac{1}{n^{\sqrt{2}}}$$

(iv) $$\sum_{n=1}^{\infty} n^{0.001}$$

2. If the series has the form $\sum ar^{n-1}$ or $\sum ar^n$, it is a geometric sereis, which converges if $|r| < 1$ and diverges if $|r| \geq 1$. Some preliminary algebraic manipulation by be required to bring the series into this form.

Example. Determine the convergence or divergence of the following series.

(i) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3^n}$

(ii) $\displaystyle\sum_{n=1}^{\infty} 3^n 8^{-n-1}$

(iii) $\displaystyle\sum_{n=1}^{\infty} e^{-n}$

(iv) $\displaystyle\sum_{n=1}^{\infty} (-1)^n (0.99999)^n$

3. If the series has a form that is similar to a p-series or geometric series, then one of the comparison tests should be considered. In particular, if a_n is a rational function or algebraic function of n (involving roots of polynomials), then series should be compared with a p-series. The value of p should be chosen by keeping only the highest powers of the numerator and denominator. The comparison tests apply only to series with positive terms, but if $\sum a_n$ has some negative terms, then we can apply the Comparison Test to $\sum |a_n|$ and test for absolute convergence.

Example. Determine the convergence or divergence of the following series.

(i) $$\sum_{n=1}^{\infty} \frac{n^2 - 1}{n^2 + n}.$$

(ii) $$\sum_{n=1}^{\infty} \frac{n^2 - 1}{n^2 + n}.$$

(iii) $$\sum_{n=1}^{\infty} \frac{n}{n^4 + n^2 + n}.$$

(iv) $$\sum_{n=1}^{\infty} \frac{\sqrt{n^2 - 1}}{n^3 + 2n^2 + 5}.$$

4. If you can see at a glance that $\lim_{n\to\infty} a_n \neq 0$, then the Test for Divergence should be used.

5. If the series is of the form $\sum(-1)^n b_n$ or $\sum(-1)^{n-1} b_n$, then the Alternating Series Test is an obvious possibility.

6. Series that involve factorials or other products (including nth powers) are often conveniently tested using the Ratio Rest. Bear in mind that $\left|\frac{a_{n+1}}{a_n}\right| \to 1$ as $n \to \infty$ for all p-series and therefore all rational or algebraic functions of n. Thus the Ratio Test should *not* be used for such series.

Example. Test the series for convergence: $\displaystyle\sum_{n=1}^{\infty} \left(\frac{3n}{1+8n}\right)^n$

7. If $a_n = f(n)$, where $\int_1^\infty f(x)\,dx$ is easily evaluated, then the Integral Test is effective (assuming the hypotheses of this test are satisfied).

Example. Determine the convergence of $\sum_{k=1}^{\infty} k^2 e^{-k^3}$.

Advanced Strategies

Example. Test the series for convergence: $\sum_{n=1}^{\infty} \frac{\cos(n/2)}{n^2 + 4n}$.

Crude Estimates

- $|\sin(x)| \leq 1$.

- $|\cos(x)| \leq 1$.

- $|\arctan(x)| \leq \frac{\pi}{2}$.

Logarithms

When dealing with logarithms, remember that they grow more slowly than any power function:

$$\lim_{x \to \infty} \frac{\ln x}{x^p} = 0,$$

as long as $p > 0$. This means for all p, $\frac{\ln x}{x^p}$ is eventually less than one. In other words,

$$\log x \le x^p \qquad \text{(for } x \text{ big enough).}$$

Example. Determine the convergence of $\displaystyle\sum_{k=1}^{\infty} \frac{k \ln k}{(k+1)^3}$.

Exponentials

Exponentials grow faster than any power function. Meaning,

$$\lim_{n \to \infty} \frac{n^p}{e^n} = 0,$$

for all $p > 0$ (why?). So as long as n is big enough, we have

$$\frac{n^p}{e^n} < 1 \implies e^{-n} < \frac{1}{n^p}.$$

Example. Determine the convergence of $\sum_{k=1}^{\infty} k^5 e^{-k^3}$.

Using Taylor Series to decide convergence

Example. Test the series for convergence:

(i) $\displaystyle\sum_{n=1}^{\infty} \frac{\sin(1/n)}{n}$

(ii) $\displaystyle\sum_{n=1}^{\infty} \tan(1/n)$

(iii) $\displaystyle\sum_{n=1}^{\infty} \frac{\tan(1/n)}{n}$

Theory of Convergence

Midterm 2, Problem 2

Suppose $\sum_{n=1}^{\infty} a_n = 0.6$ and $a_n > 0$ for all n. Let $s_n = a_1 + a_2 + \cdots + a_n$. Which of the following statements must be true, which must be false, and for which is it impossible to decide? Explain your reasoning.

(a) $\lim_{n \to \infty} s_n = 0$

(b) $a_{n+1} < a_n$ for all n

(c) $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} < 1$

(d) $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} > 1$

(e) $\sum_{n=1}^{\infty} s_n$ converges

(f) $\sum_{n=1}^{\infty} a_n^2$ converges

(g) $\sum_{n=1}^{\infty} \ln a_n$ converges

(h) $\sum_{n=1}^{\infty} \ln(1 + a_n)$ converges

Power Series

Recognizing a series as the evaluation of a power series

Suppose the series $\sum_{n=1}^{\infty}(x-2)^n b_n$ converges at $x = 4$ and diverges at $x = 0$.

What is the interval of convergence?

(Continued) Determine whether the following series converge:

(i) $\sum_{n=1}^{\infty} (-1)^n 2^n b_n$

(ii) $\sum_{n=1}^{\infty} b_n$

(iii) $\sum_{n=1}^{\infty} \frac{1}{b_n}$

Computing Taylor Series

Famous Series

- $e^u = 1 + u + \frac{1}{2!}u^2 + \frac{1}{3!}u^3 + \cdots = \sum_{n=0}^{\infty} \frac{u^n}{n!}$

- $\sin u = u - \frac{1}{3!}u^3 + \frac{1}{5!}u^5 - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$

- $\cos u = 1 - \frac{1}{2!}u^2 + \frac{1}{4!}u^5 - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$

- $\frac{1}{1-u} = 1 + u + u^2 + u^3 + \cdots = \sum_{n=0}^{\infty} u^n$

Example. (a) Find a power series representation of $f(x) = e^{-x^2}$

(b) Use it to find a series converging to $\int_0^\infty e^{-x^2}\,dx$.

Example. Let $f(x) = x^3 \arctan x^2$. Find $f^{(100)}(0)$.

Rates of Convergence
or, the dreaded Remainder Estimate

Shannon and Sayid are castaways on a tropical island. To pass the time, Shannon has baked a coconut cream pie. She tells Sayid she will give him a piece if he can compute $\ln 2$ and guarantee his error is no more than 10^{-10}.

Sayid has no calculator (it got destroyed in the plane crash) but is confident because he remembers his Math 1b very well. He finds the power series for $\ln(1+x)$ and plugs in $x = 1$.

Find the series Sayid claims converges to $\ln 2$.

“Wait,” says Shannon. “Are you sure that series converges?”

Find the radius of convergence of the power series in question.

“Okay,” says Sayid, “maybe we don’t know for sure. But I know the first few digits by heart and the series seems to converge to the right thing.”

Assume Sayid adds terms together at the rate of one term every second (He got an A in Math 1b). How long before Sayid can guarantee he is within 10^{-10} of $\ln 2$?

“I think I’ll go see if Sawyer wants a piece,” says Shannon.

“Hold on!” says Sayid. “I have another idea.” He realized that

$$\ln 2 = 2\ln\left(\frac{3}{2}\right) - \ln\left(\frac{9}{8}\right)$$

Find Sayid’s second series that converges to $\ln 2$.

Now how long before Sayid is eating pie?